CPSC 351 — Tutorial Exercise #17

Conditional Probability and Independence

These problems are intended to help you review ***conditional probability*** and the ***independence*** of events.

Problems To Be Solved

1. Consider an experiment in which coin tosses are used to generate sequences of six coin tosses — so that

   $$\Omega = \{\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \rangle \mid \alpha_i \in \{\mathtt{H}, \mathtt{T}\} \text{ for } 1 \leq i \leq 6\},$$

   and $|\Omega| = 2^6 = 64$.

   Four of these coin tosses are generated using a fair coin. In particular, for each choice of values $\beta_1, \beta_2, \beta_4, \beta_5 \in \{\mathtt{H}, \mathtt{T}\}$,

   $$\mathsf{P}(\alpha_1 = \beta_1 \text{ and } \alpha_2 = \beta_2 \text{ and } \alpha_4 = \beta_4 \text{ and } \alpha_5 = \beta_5) = 2^{-4} = \tfrac{1}{16}.$$

   However, values for α_3 and α_6 are determined by the other values:

   $$\alpha_3 = \begin{cases} \mathtt{H} & \text{if } \alpha_1 = \alpha_2, \\ \mathtt{T} & \text{if } \alpha_1 \neq \alpha_2 \end{cases} \qquad \text{and} \qquad \alpha_6 = \begin{cases} \mathtt{H} & \text{if } \alpha_4 = \alpha_5, \\ \mathtt{T} & \text{if } \alpha_4 \neq \alpha_5. \end{cases}$$

   This can be used to argue that *sixteen* of the outcomes in ω each have probability $\frac{1}{16}$, while all the rest have probability 0. In particular, for each set of values $\beta_1, \beta_2, \beta_3, \beta_4 \in \{\mathtt{H}, \mathtt{T}\}$, let

   $$S_{\beta_1,\beta_2,\beta_3,\beta_4} = \{\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \rangle \in \Omega \mid \alpha_1 = \beta_1, \alpha_2 = \beta_2, \alpha_4 = \beta_3, \text{ and } \alpha_5 = \beta_4\}.$$

   Each of these events includes one outcome that has probability $\frac{1}{16}$ and another three outcomes that have probability zero.

It will be helpful, when answering this question, to note that these sets are ***pairwise disjoint***.

For each of the integers i such that $1 \leq i \leq 6$, let H_i be the event "$\alpha_i = \mathrm{H}$".

(a) Prove that $\mathsf{P}(H_i) = \frac{1}{2}$ for every integer i such that $1 \leq i \leq 6$.

(b) Prove that the events H_1, H_2, H_3, H_4, H_5 and H_6 are ***pairwise independent***.

(c) Prove that the events H_1, H_2, H_3, H_4, H_5 and H_6 are ***not mutually independent***.