Lecture #21: Tail Bounds
Key Concepts

Basic Bounds

Theorem (Basic Inequality)**.** *Let Ω be a* **finite** *sample space with probability distribution P : $\Omega \to \mathbb{R}$, let $X : \Omega \to \mathbb{R}$ be a random variable, and let $h : \mathbb{R} \to \mathbb{R}$ be a total function such that*

$$h(x) \geq 0 \quad \textit{for all } x \in \mathbb{R}.$$

Then, for every real number a such that $a > 0$,

$$\mathsf{P}(h(X) \geq a) \leq \frac{\mathsf{E}[h(X)]}{a}.$$

Corollary (Markov's Inequality)**.** *Let Ω be a* **finite** *sample space with probability distribution P : $\Omega \to \mathbb{R}$, and let $X : \Omega \to \mathbb{R}$ be a random variable. Then, for every* **positive** *real number a,*

$$\mathsf{P}(|X| \geq a) \leq \frac{\mathsf{E}[|X|]}{a}.$$

Variance and Standard Deviation

Definition. Let Ω be a sample space with probability distribution P : $\Omega \to \mathbb{R}$, and let $X : \Omega \to \mathbb{R}$. Then the ***variance*** of X, with respect to P, is

$$\mathsf{var}(X) = \sum_{\mu \in \Omega} (X(\mu) - \mathsf{E}[X])^2 \times \mathsf{P}(\mu)$$

and the ***standard deviation*** of X, denoted $\sigma(X)$, is $\sqrt{\mathsf{var}(X)}$.

Theorem. *Let Ω be a* **finite** *sample space, let P : $\Omega \to \mathbb{R}$ be a probability distribution for Ω, and let X be a random variable. Then X^2 is also a random variable, and*

$$\mathsf{var}(X) = \mathsf{E}[X^2] - \mathsf{E}[X]^2.$$

It is *not* generally true that var$(X+Y) = $ var$(X) + $ var(Y) for a pair of random variables X and Y — so the following result is useful (and not trivial):

Theorem. *Let Ω be a* **finite** *sample space with probability distribution $P : \Omega \to \mathbb{R}$ and let $X_1, X_2, \ldots, X_n : \Omega \to \mathbb{R}$ be random variables (for some positive integer n). If $X_1, X_2, \ldots, X_n$ are* **pairwise independent** *then*

$$\mathrm{var}(X_1 + X_2 + \cdots + X_n) = \mathrm{var}(X_1) + \mathrm{var}(X_2) + \cdots + \mathrm{var}(X_n).$$

More Bounds

Theorem (Chebyshev's Inequality)**.** *Let Ω be a* **finite** *sample space with probability distribution $P : \Omega \to \mathbb{R}$, let X be a random variable, and let $a \in \mathbb{R}$ such that $a > 0$. Then*

$$\mathsf{P}(|X| \geq a) \leq \frac{\mathsf{E}[X^2]}{a^2}.$$

Theorem. *Let Σ be a* **finite** *sample space with probability distribution $P : \Omega \to \mathbb{R}$, let $X : \Omega \to \mathbb{R}$ be a random variable, and let $a \in \mathbb{R}$ such that $a > 0$. Then*

$$\mathsf{P}(X - \mathsf{E}[X] \geq a) \leq \frac{\mathrm{var}(X)}{a^2 + \mathrm{var}(X)}.$$

Cantelli's Inequality is sometimes called the "One-Sided Chebyshev's Inequality".

Theorem (Chernoff Bound)**.** *Let Ω be a finite sample space with probability distribution $\mathsf{P} : \Omega \to \mathbb{R}$. Suppose that $X_1, X_2, \ldots, X_n$ are mutually independent random variables such that $X_i : \Omega \to \{0, 1\}$ for $1 \leq i \leq n$, and suppose that $\mathsf{P}(X_i = 1) = p$ for every integer i such that $1 \leq i \leq n$, for a real number p such that $0 \leq p \leq 1$. Let $X = X_1 + X_2 + \cdots + X_n$. Then, for every real number θ such that $0 \leq \theta \leq 1$,*

$$\mathsf{P}(X \geq (1+\theta)pn) \leq e^{-\frac{\theta^2}{3}pn}.$$