Lecture #19: Conditional Probability and Independence
Key Concepts

Almost everything here should a part of a ***review*** of material from a prerequisite course — which might have presented the material somewhat differently.

Conditional Probability

Let Ω be a sample space and let $A, B \subseteq \Omega$ be events such that $\mathsf{P}(B) > 0$. The ***conditional probability*** of A given B, denoted $\mathsf{P}(A \mid B)$, is

$$\mathsf{P}(A \mid B) = \frac{\mathsf{P}(A \cap B)}{\mathsf{P}(B)}.$$

$\mathsf{P}(A \mid B)$ is not defined if $\mathsf{P}(B) = 0$.

Several results, that can be useful when you need to compute probabilities, were introduced:

Claim (Law of Total Probability)**.** *Let Ω be a sample space and let $P : \Omega \to \mathbb{R}$ be a probability distribution. Then, for any events A and B.*

$$\mathsf{P}(A) = \mathsf{P}(A \mid B) \cdot \mathsf{P}(B) + \mathsf{P}(A \mid \overline{B}) \cdot \mathsf{P}(\overline{B}).$$

Claim (Extended Partition Rule)**.** *Let Ω be a sample space, let P : $\Omega \to \mathbb{R}$ be a probability distribution, let k be a positive integer, and let A and $B_1, B_2, \dots, B_k$ be events satisfying the following properties.*

(a) $B_1, B_2, \dots, B_k$ are pairwise disjoint. *That is, $B_i \cap B_j = \emptyset$ for all integers i and j such that $1 \leq i, j \leq k$ and $i \neq j$.*

(b) $A \subseteq B_1 \cup B_2 \cup \dots \cup B_k$.

Then

$$\mathsf{P}(A) = \mathsf{P}(A \mid B_1) \cdot \mathsf{P}(B_1) + \mathsf{P}(A \mid B_2) \cdot \mathsf{P}(B_2) + \dots + \mathsf{P}(A \mid B_k) \cdot \mathsf{P}(B_k).$$

Claim (Baye's Theorem)**.** *Let Ω be a sample space, let P : $\Omega \to \mathbb{R}$, and let A and B be events such that $\mathsf{P}(A) > 0$ and $\mathsf{P}(B) > 0$. Then*

$$\mathsf{P}(A \mid B) = \frac{\mathsf{P}(B \mid A) \cdot \mathsf{P}(A)}{\mathsf{P}(B)}.$$

Independence

Events A and B are said to be ***independent*** if

$$\mathsf{P}(A \cap B) = \mathsf{P}(A) \times \mathsf{P}(B).$$

This can be generalized in two — different — ways when three or more random variables are being considered:

Let Ω be a sample space with probability distribution $\mathsf{P} : \Omega \to \mathbb{R}$. Let $A_1, A_2, \ldots, A_k \subseteq \Omega$ be events, for some integer $k \geq 2$. The events $A_1, A_2, \ldots, A_k$ are ***mutually independent*** if

$$\mathsf{P}\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} \mathsf{P}(A_i) \tag{1}$$

for *every* S of $\{1, 2, \ldots, k\}$.

Note: The condition at line (1) is guaranteed to hold whenever $|S| \leq 1$, so this condition only needs to be considered when $|S| \geq 2$.

The events $A_1, A_2, \ldots, A_k$ are ***pairwise independent*** if

$$\mathsf{P}(A_i \cap A_j) = \mathsf{P}(A_i) \times \mathsf{P}(A_j) \tag{2}$$

for every pair of integers i and j such that $1 \leq i, j \leq k$ and $i \neq j$.

If events $A_1, A_2, \ldots, A_k$ are ***mutually independent*** then they are also ***pairwise independent***. The converse does not always hold.