Lecture #18: Probability Distributions
Key Concepts

Almost everything here should a part of a ***review*** of material from a prerequisite course — which might have presented the material somewhat differently.

Experiments and Sample Spaces

An ***experiment*** is a procedure (or process) that yields one of a given set of possible ***outcomes***. The set of possible outcomes of the experiment — which we will often name Ω — is called the ***sample space***. In this course we will (almost always) consider experiments where the sample space, Ω, is countable — studying a part of probability theory called ***discrete probability theory***.

Events

An ***event*** is a subset of the experiment's sample space Ω. Events are used to model the kinds of "things that are interested in" that will be studied.

An ***elementary event*** is a set of size one — that is, it is an event that only includes a single outcome. (Some references might use the name "outcome" for these too.)

Probability Distributions

Consider an experiment with sample space Ω. A ***probability distribution*** is a (total) function $\mathsf{P} : \Omega \to \mathbb{R}$ such that $0 \leq \mathsf{P}(x) \leq 1$ for every outcome $x \in \Omega$, and such that

$$\sum_{x \in \Omega} \mathsf{P}(x) = 1.$$

For any set Ω, let $\mathcal{P}(\Omega)$ denote the set of all ***subsets*** of Ω. A probability distribution P (on an experiment with a countable sample space) is "extended" to get a function

$$\mathsf{P} : \mathcal{P}(\Omega) \to \mathbb{R}$$

by setting

$$\mathsf{P}(A) = \sum_{x \in A} \mathsf{P}(x)$$

for every event $A \subseteq \Omega$ (that is, for all $A \in \mathcal{P}(\Omega)$).

Uniform Distributions

If Ω is a finite set then the ***uniform probability distribution*** (for Ω) defines the probability of every outcome to be the same: This is the function $\mathsf{P} : \Omega \to \mathbb{R}$ such that

$$\mathsf{P}(x) = \frac{1}{|\Omega|}$$

for every outcome $x \in \Omega$.

Suppose that $A \subseteq \Omega$ is an event. Then, if P is the uniform distribution for Ω, then

$$\begin{aligned}
\mathsf{P}(A) &= \sum_{x \in A} \mathsf{P}(x) \\
&= \sum_{x \in A} \tfrac{1}{|\Omega|} \\
&= \tfrac{1}{|\Omega|} \sum_{x \in A} 1 \\
&= \tfrac{1}{|\Omega|} \cdot |A| \\
&= \frac{|A|}{|\Omega|}.
\end{aligned}$$

Probability of the Complement of an Event

If Ω is a sample space for an experiment and $A \subseteq \Omega$ is an event, then the ***complement***[1] of the event A, $\overline{A}$, is the set of outcomes that *are not* in A.

$$\overline{A} = \{x \in \Omega \mid x \notin A\}.$$

Theorem 1. *Let Ω be a sample space with probability distribution P : $\Omega \to \mathbb{R}$, and let $A \subseteq \Omega$. Then the probability of the complement, $\overline{A}$, of the event A is*

$$\mathsf{P}(\overline{A}) = 1 - \mathsf{P}(A).$$

Probability of the Union of Events

Theorem 2. *Let Ω be a sample space with probability distribution P : $\Omega \to \mathbb{R}$. Then, for any events $A, B \subseteq \Omega$,*

$$\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B).$$

Corollary 3. *Let Ω be a sample space with probability distribution P : $\Omega \to \mathbb{R}$. Then, for any events $A, B \subseteq \Omega$,*

$$\mathsf{P}(A \cup B) \leq \mathsf{P}(A) + \mathsf{P}(B).$$

Theorem 4 (Union Bound)**.** *Let Ω be a sample space with probability distribution P : $\Omega \to \mathbb{R}$, let k be a positive integer, and let $E_1, E_2, \ldots, E_k \subseteq \Omega$. Then*

$$\mathsf{P}(E_1 \cup E_2 \cup \cdots \cup E_k) \leq \sum_{i=1}^{k} \mathsf{P}(E_i).$$

[1] It is also OK if you use A^C to represent the complement of A, as we did for *languages*. These are both commonly used as the name for this set.