

Lecture #15: Many-One Reductions

What Will Happen During the Lecture

Goals of this lecture presentation will be to help students understand **many-one reductions**, how to show that these exist, and how to use these reductions to prove that languages are undecidable — or unrecognizable.

Problems To Be Solved

1. Consider the alphabet

$$\Sigma_D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

Let $\sigma_0 = 0$, $\sigma_1 = 1$, $\sigma_2 = 2$, $\sigma_3 = 3$, and so on (so that, for each integer i such that $0 \leq i \leq 9$, σ_i is the symbol in Σ_D that we generally use to represent i). Recall that, for every **positive** integer n , there is an **unpadded decimal representation** of n , consisting of an integer $k \geq 0$ and digits $d_0, d_1, \dots, d_k \in \{d \in \mathbb{N} \mid 0 \leq d \leq 9\}$ such that $d_k \geq 1$ and

$$n = \sum_{\ell=0}^k d_\ell \times 10^\ell.$$

The **unpadded decimal representation** of n is the string $\sigma_{d_k} \sigma_{d_{k-1}} \dots \sigma_{d_1} \sigma_{d_0}$, a string in Σ_D^* with length $k + 1$.

For example, if $n = 312$ then the **unpadded decimal representation** of n consists of the integer $k = 2$ and the digits $d_0 = 2$, $d_1 = 1$ and $d_2 = 3$ — since $d_k = d_2 \geq 1$ — and

$$312 = 2 \times 10^0 + 1 \times 10^1 + 3 \times 10^2$$

— and the **unpadded decimal representation** of n is the string 312 — a string in Σ_D^* with length $k + 1 = 3$.¹

¹Describing this can be tricky, because we are so used to working with the “unpadded decimal representations” of positive integers that it might not be easy to see the difference between the unpadded decimal representation of the positive integer n — which is a non-empty string in Σ_D^* — and the integer n , itself.

In order to have “unpadded decimal representations” of all natural numbers — including zero — we generally define the **unpadded decimal representation** of zero to be the string 0 — a string in Σ_D^* with length one.

Let $\text{Unpad} \subseteq \Sigma_D^*$ be the set of unpadded decimal representations of natural numbers. This is certainly a decidable language — indeed, it is even a **regular** language — because this is just the union of the set of all non-empty strings in Σ_D^* that do not begin with 0, and the set $\{0\}$.

A (possibly) **padded decimal representation** of a natural number n is any string in Σ_D^* that is the concatenation of a string of zero or more 0's and the unpadded decimal representation of n . Thus the **padded decimal representations** of zero include the strings 0^i for every positive integer i , and the **padded decimal representations** of the natural number 312 are the strings of the form 0^j312 for every non-negative integer j .

Let $\text{Pad} \subseteq \Sigma_D^*$ be the set of (possibly) padded decimal representations of natural numbers. This is also certainly a decidable language — indeed, it is even a **regular** language — because this is just $\Sigma_D^+ = \Sigma_D^* \setminus \{\lambda\}$, that is, the set of all **non-empty** strings in Σ_D^* .

Now let $S \subseteq \mathbb{N}$. This can be used to define **two** languages, that are each subsets of Σ_D^* :

- $\mathcal{U}_S \subseteq \text{Unpad}$ is the set of all **unpadded decimal representations** of numbers $n \in S$.
- $\mathcal{P}_S \subseteq \text{Pad}$ is the set of all **padded decimal representations** of numbers $n \in S$.

For example, if $S_1 = \emptyset \subseteq \mathbb{N}$ then $\mathcal{U}_{S_1} = \mathcal{P}_{S_1} = \emptyset \subseteq \Sigma_D^*$. On the other hand, if $S_2 = \{312\} \subseteq \mathbb{N}$ then $\mathcal{U}_{S_2} = \{312\} \subseteq \text{Unpad}$ and $\mathcal{P}_{S_2} = \{0^j312 \mid j \in \mathbb{N}\} \subseteq \text{Pad}$.

We will prove that $\mathcal{U}_S \preceq_M \mathcal{P}_S$ (for every subset S of $\mathbb{N}$).

2. The lecture notes introduced a language $\text{HALT}_{\text{TM}} \subseteq \text{TM+I}$ and a proof that

$$\text{HALT}_{\text{TM}} \preceq_M \text{A}_{\text{TM}}.$$

We will consider at least one attempt to prove that A_{TM} is many-one reducible to HALT_{TM} as well — in order to discuss at least one **mistake that many students make**, when they try to give many-one reductions, and which you should watch for and avoid.