Lecture #13: Universal Turing Machines
Key Concepts

Following the introduction of a minor additional condition — we will no longer consider Turing machines

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}}) \tag{1}$$

such that either $q_0 = q_{\text{accept}}$ or $q_0 = q_{\text{reject}}$ — an ***encoding*** of a (standard) Turing machine M as a string of symbols over an alphabet

$$\Sigma_{\text{TM}} = \{\texttt{s}, \texttt{q}, \texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}, \texttt{L}, \texttt{R}, \texttt{Y}, \texttt{N}, \texttt{(}, \texttt{)}, \texttt{,}, \texttt{;}\}$$

was introduced. Encodings of Turing machines M (with components as at line (1), above) and input strings $\omega \in \Sigma^\star$ were introduced, as well.

Three languages were introduced:

- $L_{\text{TM}} \subseteq \Sigma_{\text{TM}}^\star$ is the language of encodings of Turing machines.
- $L_{\text{TM+I}} \subseteq \Sigma_{\text{TM}}^\star$ is the language of encodings of Turing machines M (with some input alphabet Σ) and input strings $\omega \in \Sigma^\star$.
- $A_{\text{TM}} \subseteq L_{\text{TM+I}}$ is the language of encodings of Turing machines M (with some input alphabet Σ) and input strings $\omega \in \Sigma^\star$ such that M ***accepts*** ω.

The languages L_{TM} and $L_{\text{TM+I}}$ are both ***decidable***.

A ***universal Turing machine*** is a (single tape, or multi-tape) Turing machine M_{UTM}, with input alphabet Σ_{TM}, which satisfies the following properties:

- If $\mu \in \Sigma_{\text{TM}}^\star$ and $\omega \notin L_{\text{TM+I}}$ — so that ω is *not* the encoding of any Turing machine M and input string ω for it — then M_{UTM} ***rejects*** μ.

Suppose, instead, that $\mu \in L_{\mathsf{TM+I}}$ — so that μ is the encoding of some Turing machine M (with an input alphabet Σ) and input string ω for M.

- If M ***accepts*** μ then M_{UTM} ***accepts*** the string, μ, that encodes M and ω.
- If M ***rejects*** μ then M_{UTM} ***rejects*** the string, μ. that encodes M and ω.
- If M ***loops*** on ω then M_{UTM} ***loops*** on the string, μ, that encodes M and ω.

Universal Turings exist — and one such (multi-tape) Turing machine was described in the preparatory material for this lecture.

Since A_{TM} is the language of any universal Turing machine, it follows (by the existence of these Turing machines) that the language, A_{TM}, is ***recognizable***.

However, it is possible to prove (by contradiction) that the language A_{TM} is ***undecidable***.

Furthermore, another proof by contradiction can be given to prove that the language

$$A_{\mathsf{TM}}^{C} = \{\mu \in \Sigma_{\mathsf{TM}}^{\star} \mid \mu \notin A_{\mathsf{TM}}\}$$

is ***unrecognizable***.