Lecture #13: Universal Turing Machines
Lecture Presentation

Review of Preparatory Material

Proving That A_{TM} is Undecidable

Claim. *The language A_{TM} is undecidable.*

How To Prove This:

```
On input μ ∈ Σ*_TM:
1.  if (μ ∈ L_TM) {
      Let M = (Q, Σ, Γ, δ, q_0, q_accept, q_reject) be the Turing machine that is en-
      coded by μ.
2.    if (Σ = Σ_TM) {
3.      if (M accepts μ) {    // Does M accepts its own encoding?
4.        reject μ
        } else {
5.        accept μ
        }
      } else }
6.      reject μ
      }
    } else {
7.    reject μ
    }
```

Figure 1: Algorithm Used to Get a Contradiction

Details of Proof:

Suppose, to obtain a contradiction, that the language A_{TM} is decidable. Consider the algorithm shown in Figure 1.

Subclaim: There exists a Turing machine M_D that implements the algorithm in Figure 1.

Explanation:

It follows that there exists a string $\mu_D \in \Sigma_{\mathsf{TM}}^\star$ that encodes the Turing machine M_D.

What Happens When M_D is Executed on its Encoding, μ_D?

Proving that A_{TM}^C is Unrecognizable

Claim. *The language A_{TM}^C is unrecognizable.*

How To Prove This:

```
On input $\mu \in \Sigma_{\mathsf{TM}}^{\star}$:
integer $i$ := $1$
while (true) {
  if ($M_{\mathsf{Yes}}$ accepts $\mu$ after taking at most $i$ steps) {
    accept $\mu$
  } else if ($M_{\mathsf{Yes}}$ rejects $\mu$ after taking at most $i$ steps) {
    reject $\mu$
  }
  if ($M_{\mathsf{No}}$ accepts $\mu$ after taking at most $i$ steps) {
    reject $\mu$
  } else if ($M_{\mathsf{No}}$ rejects $\mu$ after taking at most $i$ steps) {
    accept $\mu$
  }
  $i$ := $i + 1$
}
```

Figure 2: Another Algorithm Used to Get a Contradiction

Details of Proof:

Suppose to obtain a contradiction, that A_{TM}^{C} is recognizable.

- Since A_{TM} is recognizable (because it is the language of a universal Turing machine) there exists a (standard) Turing machine M_{Yes} such that $L(M) = A_{\mathsf{TM}}$.
- Since A_{TM}^{C} is also recognizable (by the above assumption) there is also a (standard) Turing machine M_{No} such that $L(M) = A_{\mathsf{TM}}^{C}$.

Consider the algorithm shown in Figure 2.

What Happens When This Algorithm is Executed on μ, when $\mu \in A_{\mathsf{TM}}$?

What Happens When This Algorithm is Executed on μ, when $\mu \notin A_{\mathsf{TM}}$?

What Can We Conclude From This?

What Just Happened?