CPSC 351 — Tutorial Exercise #10
Introduction to Turing Machines

About This Exercise

This exercise is intended to help you to learn how Turing machine processes strings over an alphabet — and how to (using a sequence of carefully chosen steps) prove that a given Turing machine ***recognizes*** — or ***decides*** a given language. It continues the problem (which introduced some aspects of "Turing machine design" introduced near the end of the presentation for Lecture #10.

A Language, Algorithm, and Turing Machine

Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and consider the language

$$\widehat{L} = \{\mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N}\}$$

— where $\mathbb{N}$ represents the set of *non-negative* integers, so that $0 \in \mathbb{N}$ and $\lambda \in \widehat{L}$.

During the lecture presentation #10, the algorithm shown in Figure 1 on page 2 was introduced and a proof of the following claim was sketched:

Claim. *Let $\omega \in \Sigma^\star$. If the algorithm* `isInL` *is executed on input ω then the execution halts after a finite number of executions of the steps shown in the algorithm. Furthermore, the algorithm returns* `true` *if $\omega \in \widehat{L}$ and the algorithm returns* `false` *if $\omega \notin \widehat{L}$.*

Consider the Turing machine

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$$

such that $Q = \{q_0, q_1, q_2, q_3, q_4, q_{\text{accept}}, q_{\text{reject}}\}$, $\Sigma = \{\mathtt{ab}\}$ (as above), $\Gamma = \Sigma \cup \{\sqcup\}$, and transitions are as shown in Figure 2 on page 3. In this diagram the accepting state is drawn as "q_A" instead of "q_{accept}" in order to keep the picture simpler.Transitions to the rejecting state are not shown — but each leaves the symbol on the tape unchanged, moving right. Thus $\delta(q_0, \mathtt{b}) = (q_{\text{reject}}, \mathtt{b}, \mathtt{R})$, $\delta(q_1, \sqcup) = (q_{\text{reject}}, \sqcup, \mathtt{R})$, $\delta(q_3, \mathtt{a}) = (q_{\text{reject}}, \mathtt{a}, \mathtt{R})$, and $\delta(q_3, \sqcup) = (q_{\text{reject}}, \sqcup, \mathtt{R})$.

```
boolean isInL ( ω : Σ* ) {
1.  if (ω == λ) {
2.    return true
3.  } else if (ω begins with "b") {
4.    return false
5.  } else if (|ω| == 1) {
6.    return false
7.  } else if (ω ends with "a") {
8.    return false
    } else {
9.    Let μ ∈ Σ* such that ω = a · μ · b. Return isInL(μ)
    }
```

Figure 1: An Algorithm to Decide Membership in $\widehat{L}$

Problems To Be Discussed in Tutorials

Problems #1–#3, below, can each be solved by giving a short ***trace of execution***, so they should not take very much time. Neither should Problem #5, if you make use of the result that is proved if you complete Problem #4.

1. Use a trace of execution to show that if M is in a configuration $\sqcup^h\, q_0$ for a non-negative integer h then M halts, in its accepting state, after one more step.

   Use this to argue that M ***accepts*** the empty string, λ.

   Note: If you completed this then you should (arguably) be able to see that q_0, and the transition out of this start state for λ, implement the test at line 1 of the "`isIn`" algorithm and the step at line 2.

2. Let $\omega \in \Sigma^\star$ such that ω begins with "b" — so that $\omega = \mathtt{b}\cdot\mu$ for a string $\mu \in \Sigma^\star$. Use a trace of execution to show that if M is in configuration $\sqcup^h\, q_0\, \omega$, for a non-negative integer h, then M halts, in its rejecting state, after taking a finite number of additional steps.

   Use this to argue that M ***rejects*** every string in $\Sigma^\star$ that begins with "b".

   Note: If you completed this then you should (arguably) be able to see that q_0, and the transition out of this start state for b, implement the test at line 3 of the "`isIn`" algorithm and the step at line 4.

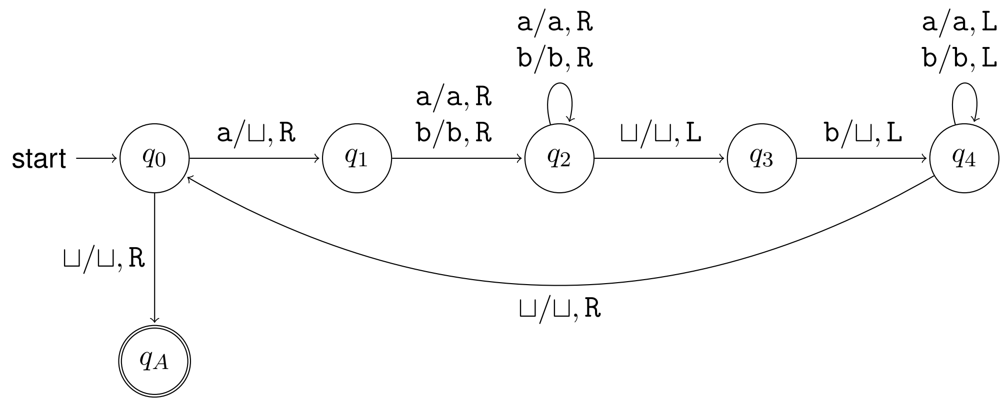

Figure 2: A Turing Machine That Decides $\widehat{L}$

3. Use a trace of execution to show that if M is in configuration $\sqcup^h\, q_0\, \mathtt{a}$, for a non-negative integer h, then M halts, in its rejecting state, after taking a finite number of additional steps.

   Use this to argue that M ***rejects*** the string "a".

   Note: The only way for the test at line 5 of the "`isIn`" algorithm to be reached, and passed, is for the input string to be the string "a". Thus, if you completed this, then you should (arguably) be able to show that the transition for a out of state q_0, the state q_1, and the transition for $\sqcup$ out of q_1, implement the test at line 5 of the algorithm and the step at line 6.

4. Consider, now, a string

$$\omega = \mathtt{a}\, \alpha_1 \alpha_2 \dots \alpha_n \in \Sigma^\star$$

   where $n \geq 1$ and $\alpha_1, \alpha_2, \dots, \alpha_n$ — so that ω is a string with length at least two such that ω begins with "a".

   Prove the following.

   Claim. $\sqcup^h\, q_0\, \omega \;=\; \sqcup^h\, q_0\, \mathtt{a}\, \alpha_1\alpha_2 \dots \alpha_n \;\vdash^\star\; \sqcup^{h+1}\, \alpha_1\alpha_2 \dots \alpha_i\, q_2\, \alpha_{i+1}\alpha_{i+2} \dots \alpha_n$ *for every integer i such that $1 \leq i \leq n$.*

   Note: This asserts that $\sqcup^h\, q_0\, \omega = \sqcup^h\, q_0\, \mathtt{a} \cdot \mu \vdash^\star \sqcup^{h+1}\, \mu\, q_2$ when $i = n$.

5. Let $\omega \in \Sigma^\star$ such that $|\omega| \geq 2$ and ω begins, and ends, with "a" — so that $\omega = \mathtt{a} \cdot \mu \cdot \mathtt{a}$ for a string $\mu \in \Sigma^\star$. Let h be a non-negative integer.

   Prove that if M is in configuration $\sqcup^h\, q_0\, \omega$ then M halts, in its rejecting state, after taking a finite number of additional steps.

Use this to argue that M ***rejects*** every string in $\Sigma^\star$, with length at least two, that begins and ends with "a".

Note: This should not be very hard to do, if you use the claim that was proved just before this.

Note: The only way for the test at line 7 of the "`isIn`" algorithm to be reached, and passed, is for the input string to be a string $\omega = \mathtt{a} \cdot \mu \cdot \mathtt{a}$ for $\mu \in \Sigma^\star$, that is, a string with length at least two that begins and ends with "a". Thus, if you completed this, then you should (arguably) be able to show that states q_1, q_2 and q_3, the transitions from q_1 to q_2, from q_2 to itself, from q_2 to q_3, and the transition out of q_3 for a, implement the test at line 7 and the step at line 8.

Additional Problems

There will almost certainly *not* be time to discuss these problems in tutorial. By solving them (if you have time) you will complete a proof that M decides the language $\widehat{L}$ (at least, arguably) by implementing the algorithm "`isIn`".

6. Use a trace of execution to to show that if M is in configuration $\sqcup^h\, q_0\mathtt{ab}$ for a non-negative integer h then M goes to configuration $\sqcup^{h+1}\, q_0$ after a finite number of steps.

7. Let h be a non-negative integer and let $\mu \in \Sigma^\star$ such that $|\mu| \geq 1$. Prove that

$$\sqcup^{h+1}\, \mu\, q_3\, \mathtt{b} \vdash^\star \sqcup^h\, q_4 \sqcup\, \mu$$

   for every non-negative integer h and for every string $\mu \in \Sigma^\star$.

   Note: Suppose that $\mu = \alpha_n\alpha_{n-1}\ldots\alpha_1$ for some positive integer n and for symbols $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Sigma = \{\mathtt{a}, \mathtt{b}\}$. Before trying to prove the above result, you should consider stating, and proving, a different ***claim*** — somewhat resembling the claim that was proved when solving Problem #4 — that concerns what configurations that are reached as the Turing machine starts in configuration $\sqcup^{h+1}\, \mu\, q_3\, \mathtt{b}$ and moves its tape head back to the left (while in state q_4).

   The proof — and the situations you must consider — probably *will* be less confusing if you write μ as $\alpha_n\alpha_{n-1}\ldots\alpha_1$, as above, instead of "$\alpha_1\alpha_2\ldots\alpha_n$".

8. Use the above results to prove that if h is a non-negative integer and $\mu \in \Sigma^\star$ such that $|\mu| \geq 1$ then

$$\sqcup^h\, q_0\, \mathtt{a} \cdot \mu \cdot \mathtt{b} \vdash^\star \sqcup^{h+1}\, q_0\, \mu$$

9. Use the above results to prove the following.

   Claim. *Let $h \geq 0$ and let $\omega \in \Sigma^\star$ If M is in configuration $\sqcup^h q_0 \omega$ and $\omega \in \widehat{L}$ then M moves to an accepting configuration after a finite number of steps. On the other hand, if $\omega \notin \widehat{L}$, then M moves to a rejecting configuration after a finite number of steps, instead.*

   Note that, since the accepting configuration for ω is the configuration $\sqcup^h q_0 \omega$ when $h = 0$, this claim that M ***decides*** the language L, as desired.

   Suggestion: Consider using induction on the length of ω — using the ***strong*** form of mathematical induction, with the cases $|\omega| = 0$ and $|\omega| = 1$ considered in the basis.