Lecture #10: Introduction to Turing Machines

Lecture Presentation

Review of Preparatory Material

Continuing the First Example

Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and consider, once again, a Turing machine

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$$

such that

$$Q = \{q_0, q_{\mathtt{a},1}, q_{\mathtt{a},2}, q_{\mathtt{b},1}, q_{\mathtt{b},2}, q_{\text{accept}}, q_{\text{reject}}\},$$

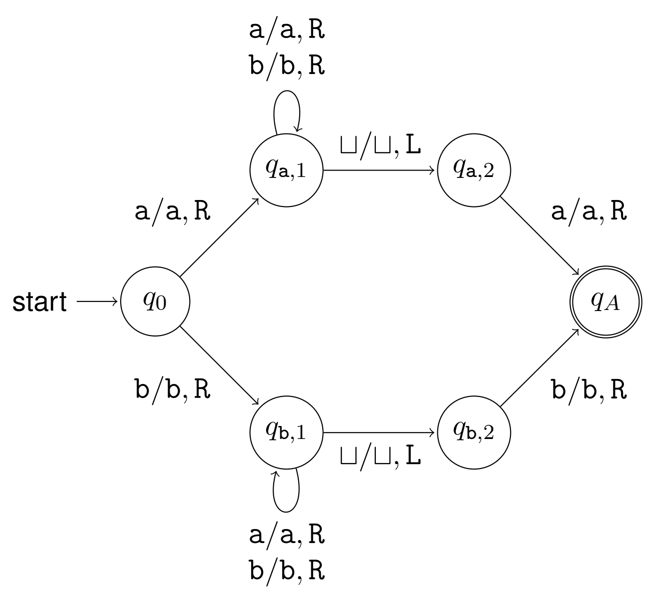
$\Gamma = \{\mathtt{a}, \mathtt{b}, \sqcup\}$, and the transition function, δ, is given by the following transition table:

	$\mathtt{a}$	$\mathtt{b}$	$\sqcup$
q_0	$(q_{\mathtt{a},1}, \mathtt{a}, \mathrm{R})$	$(q_{\mathtt{b},1}, \mathtt{b}, \mathrm{R})$	$(q_{\text{reject}}, \sqcup, \mathrm{R})$
$q_{\mathtt{a},1}$	$(q_{\mathtt{a},1}, \mathtt{a}, \mathrm{R})$	$(q_{\mathtt{a},1}, \mathtt{b}, \mathrm{R})$	$(q_{\mathtt{a},2}, \sqcup, \mathrm{L})$
$q_{\mathtt{a},2}$	$(q_{\text{accept}}, \mathtt{a}, \mathrm{R})$	$(q_{\text{reject}}, \mathtt{b}, \mathrm{R})$	$(q_{\text{reject}}, \sqcup, \mathrm{R})$
$q_{\mathtt{b},1}$	$q_{\mathtt{b},1}, \mathtt{a}, \mathrm{R})$	$q_{\mathtt{b},1}, \mathtt{b}, \mathrm{R})$	$(q_{\mathtt{b},2}, \sqcup, \mathrm{L})$
$q_{\mathtt{b},2}$	$q_{\text{reject}}, \mathtt{a}, \mathrm{R})$	$(q_{\text{accept}}, \mathtt{b}, \mathrm{R})$	$(q_{\text{reject}}, \sqcup, \mathrm{R})$

This Turing machine can also be described using the following picture. To keep the picture simple the accepting state is shown as "q_A" instead of q_{accept}, and transitions to the rejecting state, and this state, are left out — but, as the transition table indicates,

$$\delta(q, \sigma) = (q_{\text{reject}}, \sigma, \mathrm{R})$$

for $q = q_0$ and $\sigma = \sqcup$, for $q = \mathtt{a}, 2$ and either $\sigma = \mathtt{b}$ or $\sigma = \sqcup$, and for $q_{\mathtt{b},2}$ and either $\sigma = \mathtt{a}$ or $\sigma = \sqcup$.



The goal of the first part of this lecture presentation is to sketch a proof that this Turing machine ***decides*** the language

$$\begin{aligned} L &= \{\omega \in \{\mathtt{a}, \mathtt{b}\}^\star \mid |\omega| \geq 1 \text{ and } \omega \text{ begins, and ends, with the same letter}\} \\ &= \{\mathtt{a}, \mathtt{b}\} \cup \{\mathtt{a}\mu\mathtt{a} \mid \mu \in \{\mathtt{a}, \mathtt{b}\}^\star\} \cup \{\mathtt{b}\mu\mathtt{b} \mid \mu \in \{\mathtt{a}, \mathtt{b}\}^\star\}. \end{aligned}$$

Traces of Execution on Short Strings

Give a ***trace of execution*** to show that M ***rejects*** the empty string, λ:

Trace of Execution:

Give a ***trace of execution*** to show that M ***accepts*** $\mathtt{a}$.

Trace of Execution:

Give a ***trace of execution*** to show that M ***accepts*** $\mathtt{b}$.

Trace of Execution:

Longer Strings That Begin With “a”

Consider a string

$$\omega = \mathsf{a}\,\alpha_1\alpha_2 \ldots \alpha_n \in \Sigma^\star$$

where $n \geq 1$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Sigma$ (so that $|\omega| = n+1$).

Consider the following:

Lemma 1. *For every integer i, if $0 \leq i \leq n$ then*

$$q_0\omega \vdash^\star \mathsf{a}\,\alpha_1\alpha_2 \ldots \alpha_i\, q_{\mathsf{a},1}\, \alpha_{i+1}\alpha_{i+1} \ldots \alpha_n.$$

Note: This asserts that

$$q_0\omega \vdash^\star \mathsf{a}\, q_{\mathsf{a},1}\alpha_1\alpha_2 \ldots \alpha_n$$

(because this is what is claimed for the case that $i = 0$). It also asserts that

$$q_0\omega \vdash^\star \mathsf{a}\,\alpha_1\alpha_2 \ldots, \alpha_n\, q_{\mathsf{a},1}$$

(because this is what is claimed for the case that $i = n$).

Proof of This Lemma:

Lemma 2. *Suppose that*

$$\omega = \mathtt{a}\,\mu\,\mathtt{a}$$

for a string $\mu \in \Sigma^\star$. Then M **accepts** *ω.*

Proof:

Lemma 3. *Suppose that*

$$\omega = \mathtt{a}\,\mu\,\mathtt{b}$$

for a string $\mu \in \Sigma^\star$. Then M **rejects** *ω.*

Proof:

Longer Strings That Begin with "b"

Now consider a string

$$\omega = \mathtt{b}\,\alpha_1\alpha_2\ldots\alpha_n \in \Sigma^\star$$

where $n \geq 1$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Sigma$ (so that $|\omega| = n + 1$).

Things To Do Next:

- State, and prove, another lemma, "Lemma 4", that resembles Lemma 1 — but that concerns the beginning of M's execution on a string that begins with "b" (like the above one) instead of a string that begins with "a".
- Use this to prove "Lemma 5" and Lemma 6", which are given below.

  Lemma 5. *Suppose that*

  $$\omega = \mathtt{b}\,\mu\,\mathtt{b}$$

  for a string $\mu \in \Sigma^\star$. Then M **accepts** *ω.*

  Lemma 6. *Suppose that*

  $$\omega = \mathtt{b}\,\mu\,\mathtt{a}$$

  for a string $\mu \in \Sigma^\star$. Then M **rejects** *ω.*

Completing a Proof That M Decides the Language L

Introduction to Turing Machine Design

It is recommended that, when designing a Turing machine for a given language by ***starting at a high level*** — picking an algorithm that can, eventually, by implemented using a Turing machine. It this not obvious you should continue by proving the correctness of the algorithm you will use.

With that noted, suppose, once again, that $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, and let

$$\widehat{L} = \{\mathtt{a}^n \mathtt{b}^n \mid n \in \mathbb{N}\}$$

— where $\mathbb{N}$ represents the set of *non-negative* integers, so that $0 \in \mathbb{N}$ and $\lambda \in \widehat{L}$.

Consider the following ***algorithm*** — which has a string $\omega \in \Sigma^\star$ as input:

```
boolean isInL ( ω : Σ* ) {
1.  if (ω == λ) {
2.    return true
3.  } else if (ω begins with "b") {
4.    return false
5.  } else if (|ω| == 1) {
6.    return false
7.  } else if (ω ends with "a") {
8.    return false
    } else {
9.    Let μ ∈ Σ* such that ω = a · μ · b. Return isInL(μ)
    }
```

Consider, as well, the following claim:

Claim. *Let $\omega \in \Sigma^\star$. If the algorithm* `isInL` *is executed on input ω then the execution halts after a finite number of executions of the steps shown in the algorithm. Furthermore, the algorithm returns* `true` *if $\omega \in \widehat{L}$ and the algorithm returns* `false` *if $\omega \notin \widehat{L}$.*

Case: $\omega = \lambda$

Case: $|\omega| = 1$

Case: $|\omega| \geq 2$

Subcase: $|\omega| \geq 2$ **and** ω **begins, and ends, with** "a"

Subcase: $|\omega| \geq 2$ **and** ω **begins with "**$\texttt{a}$**" and ends with "**$\texttt{b}$**"**

Subcase: $|\omega| \geq 2$ and ω begins with "$\texttt{b}$"

Based On The Above: What Proof Technique Can Be Used To Prove This Claim?

What Will Happen Next