Lecture #9: Nonregular Languages, Part Two
Key Concepts

The Big Idea

If P and Q are Boolean statements (that is, statements that are either *true* or *false*) and

$$P \implies Q$$

then

$$\neg Q \implies \neg P$$

as well. Thus the ***closure properties*** that were introduced in Lecture #7 (and others like them) — that were used to show that if *one* language was ***regular***, then *another* was regular too — can also be used that if *one* language is ***not regular***, then *another* language is not regular, either.

Closure Properties, How They Were Used Before, and How They Can Be Used Now

Closure Under Union

Theorem 1. *Suppose that $L_1, L_2 \subseteq \Sigma^\star$ for an alphabet Σ. If L_1 and L_2 are both regular languages, then $L_1 \cup L_2$ is also a regular language.*

How This Was Used Before: Show that some language $L \subseteq \Sigma^\star$is regular, by showing that $L = L_1 \cup L_2$, where $L_1, L_2 \subseteq \Sigma^\star$ are regular languages (and this has already been proved, or is being proved now).

How This Is Also Being Used Now: Show that a language $L \subseteq \Sigma^\star$ is *not* regular by using the fact that the language $\widehat{L}$ is not regular, where $\widehat{L} = L \cup \widetilde{L}$, for some other language $\widetilde{L} \subseteq \Sigma^\star$

that *is* a regular language. (If it has already been shown that $\widehat{L}$ is not a regular language then you must prove this too. Similarly, if $\widetilde{L}$ has not already been proved to *be* a regular language then you must do this as well.)

This can be used as the argument in a ***proof by contradiction*** — in which you start by assuming that L *is* a regular language, and obtain a contradiction without making any other assumptions.

Closure Under Concatenation

Theorem 2. *Suppose that $L_1, L_2 \subseteq \Sigma^\star$ for an alphabet Σ. If L_1 and L_2 are both regular languages, then $L_1 \circ L_2$ is also a regular language.*

How This Was Used Before: Show that some language $L \subseteq \Sigma^\star$ is regular, by showing that $L = L_1 \circ L_2$, where $L_1, L_2 \subseteq \Sigma^\star$ are regular languages (and this has already been proved, or is being proved now).

How This Is Also Being Used Now: Show that a language $L \subseteq \Sigma^\star$ is *not* regular by using the fact that the language $\widehat{L}$ is not regular, where *either* $\widehat{L} = L \circ \widetilde{L}$ *or* $\widehat{L} = \widetilde{L} \circ L$, for some other language $\widetilde{L} \subseteq \Sigma^\star$ that *is* a regular language. (If it has already been shown that $\widehat{L}$ is not a regular language then you must prove this too. Similarly, if $\widetilde{L}$ has not already been proved to *be* a regular language then you must do this as well.)

Once again the easiest way to use this in a proof, that is easy for someone else to read and understand, is probably to write a ***proof by contradiction*** in which you establish a contradiction after assuming that L is a regular language (without assuming anything else).

Closure Under Kleene Star

Theorem 3. *Suppose that $L \subseteq \Sigma^\star$ for an alphabet Σ. If L is a regular language, then $L^\star$ is also a regular language.*

How This Was Used Before: Show that some language $L \subseteq \Sigma^\star$ is regular, by showing that $L = \widehat{L}^\star$, where $\widehat{L} \subseteq \Sigma^\star$ is a regular language (and this has already been proved, or is being proved now).

How This is Also Being Used Now: Show that a language $L \subseteq \Sigma^\star$ is *not* regular by using the fact that the language $\widehat{L} \subseteq \Sigma^\star$ is not regular, where $\widehat{L} = L^\star$ (and it has already been proved that $\widehat{L}$ is not a regular language, or is also being proved now).

Once again, it is probably easiest to use this, in a proof that someone else can read and understand, by writing a ***proof by contradiction*** that uses the assumption that L is a regular language.

A New Closure Property: Closure Under Complementation

Recall that if $L \subseteq \Sigma^\star$, for an alphabet Σ, then the ***complement*** of L is the language[1]

$$L^C = \{\omega \in \Sigma^\star \mid \omega \notin L\}.$$

Note that $(L^C)^C = L$, for every language $L \subseteq \Sigma^\star$.

Theorem 4. *Suppose that $L \subseteq \Sigma^\star$ for an alphabet Σ. If L is a regular language, then L^C is also a regular language.*

How This Could Have Been Used, Before: Show that some language $L \subseteq \Sigma^\star$ is regular, by showing that $L = \widehat{L}^C$, where $\widehat{L} \subseteq \Sigma^\star$ is a regular language (and this has already been proved, or is being proved now).

How This is Also Being Used Now: Show that a language $L \subseteq \Sigma^\star$ is *not* regular by using the fact that the language $\widehat{L} \subseteq \Sigma^\star$ is not regular, where $\widehat{L} = L^C$ (and it has already been proved that $\widehat{L}$ is not a regular language, or is also being proved now).

Once again, it is probably easiest to use this, in a proof that someone else can read and understand, by writing a ***proof by contradiction*** that uses the assumption that L is a regular language.

More To Note

If *additional* closure properties for the set of regular languages are discovered (and proved to be correct) then this will automatically yield new ways, both prove that some languages *are* regular, and also to prove that (some other) languages are *not* regular.

This approach can also be applied to other interesting sets of languages besides the set of *regular* languages. Indeed, this will be a significant part of the middle of this course.

[1] This language is sometimes represented as $\overline{L}$, instead.