Lecture #6: Equivalence of Deterministic Finite Automata and Nondeterministic Finite Automata

Proof of Claims

This document provides a somewhat more formal description of the constructions used to prove the equivalence of deterministic finite automata and nondeterministic finite automata, described in the notes for this lecture, as well as proofs that these constructions are correct. It is ***not required reading*** but is provided for students who are interested in seeing how these claims can be proved.

Simulating a Deterministic Finite Automaton

To begin, let Σ be an alphabet, let

$$M = (Q, \Sigma, \delta, q_0, F)$$

be a ***deterministic finite automaton*** with this alphabet, and let $L \subseteq \Sigma^\star$ be the language of M. Recall that, since M is a *deterministic* finite automaton, δ is a total function with domain $Q \times \Sigma$ and range Q.

Suppose that

$$\widehat{M} = (Q, \Sigma, \widehat{\delta}, q_0, F)$$

is a ***nondeterministic finite automaton*** with the same set Q if states as M, along with the same alphabet Σ, start state $q_0 \in Q$ and set $F \subseteq Q$ of accepting states. Suppose, as well, that the function $\widehat{\delta}$ is as defined as follows: For every state $q \in Q$ and for every symbol $\sigma \in \Sigma$,

$$\widehat{\delta}(q, \sigma) = \{\delta(q, \sigma)\} \tag{1}$$

— and suppose, as well, for every state $q \in Q$, that

$$\widehat{\delta}(q, \lambda) = \emptyset. \tag{2}$$

Then $\widehat{\delta}$ is a well-defined total function with domain $Q \times \Sigma_\lambda$ and range $\mathcal{P}(Q)$ — so that $\widehat{M}$ is a well-defined nondeterministic finite automaton.

Lemma 1. *Let $\widehat{M} = (Q, \Sigma, \widehat{\delta}, q_0, F)$ be as above. Then, for every state $q \in Q$,*

$$\mathsf{Cl}_\lambda(q) = \{q\}.$$

Proof. For every state $q \in Q$, $q \in \mathit{Cl}_\lambda(q)$, since the state q can be reached from itself without following λ-transitions. Since $\widehat{M}$ does not include any λ-transitions at all, q is the *only* state in Q that can be reached from q by following λ-transitions, so that $\mathit{Cl}_\lambda(q) = \{q\}$, as claimed. □

Lemma 2. *Once again, suppose that the deterministic finite automaton $M = (Q, \Sigma, \delta, q_0, F)$ and the nondeterministic nondeterministic finite automaton $\widehat{M} = (Q, \Sigma, \widehat{\delta}, q_0, F)$ are as given above. Then, for every state $q \in Q$ and for every string $\omega \in \Sigma^\star$,*

$$\widehat{\delta}^\star(q, \omega) = \{\delta^\star(q, \omega)\}.$$

Proof. This can be proved by induction on the length of the string ω, using the standard form of mathematical induction. Since every string in $\Sigma^\star$ has length at least zero, the case that $|\omega| = 0$ (that is, $\omega = \lambda$) will be considered in the basis.

Basis: Suppose that $\omega \in \Sigma^\star$ such that $|\omega| = 0$, that is, $\omega = \lambda$. Then, it follows by the definitions of the extended transition functions $\widehat{\delta}^\star$ and $\delta^\star$ of $\widehat{M}$ and M, respectively, that, for every state $q \in Q$,

$$\begin{aligned}
\widehat{\delta}^\star(q, \omega) &= \widehat{\delta}^\star(q, \lambda) && \text{(since } \omega = \lambda\text{)} \\
&= \mathit{Cl}_\lambda(q) && \text{(by the definition of } \widehat{\delta}^\star\text{)} \\
&= \{q\} && \text{(by Lemma 1)} \\
&= \{\delta^\star(q, \lambda)\} && \text{(by the definition of } \delta^\star\text{)} \\
&= \{\delta^\star(q, \omega)\} && \text{(since } \omega = \lambda\text{)}
\end{aligned}$$

as required.

Inductive Hypothesis: Let k be an integer such that $k \geq 0$. It is necessary and sufficient to use the following

> <u>Inductive Hypothesis:</u> For very state $q \in Q$ and for every string $\mu \in \Sigma^\star$ such that $|\mu| = k$,
>
> $$\widehat{\delta}^\star(q, \mu) = \{\delta^\star(q, \mu)\}.$$

to prove the following

> <u>Inductive Claim:</u> For every state $q \in Q$ and for every string $\nu \in \Sigma^\star$ such that $|\nu| = k + 1$,
>
> $$\widehat{\delta}^\star(q, \nu) = \{\delta^\star(q, \nu)\}.$$

With that noted, let $q \in Q$ and let ν be a string in $\Sigma^\star$ such that $|\nu| = k+1$. Since $k \geq 0$, $k+1 \geq 1$, so that

$$\nu = \mu \cdot \sigma$$

for some string $\mu \in \Sigma^\star$ such that $|\mu| = k$, and for some symbol $\sigma \in \Sigma$. Since the length of μ is k, it follows by the Inductive Hypothesis that

$$\widehat{\delta}^\star(q, \mu) = \{\delta^\star(q, \mu)\}.$$

Let $\widehat{q} = \delta^\star(q, \mu)$, so that $\widehat{q} \in Q$ and $\widehat{\delta}^\star(q, \mu) = \{\widehat{q}\}$ — and so that

$$\delta^\star(q, \nu) = \delta^\star(q, \mu \cdot \sigma) = \delta(\delta^\star(q, \mu), \sigma) = \delta(\widehat{q}, \sigma)$$

by the definition of $\delta^\star$.

Now

$$\begin{aligned}
\widehat{\delta}^\star(q, \nu) &= \widehat{\delta}^\star(q, \mu \cdot \sigma) && \text{(since } \nu = \mu \cdot \sigma\text{)} \\
&= \bigcup_{r \in \widehat{\delta}^\star(q,\mu)} \left(\bigcup_{s \in \widehat{\delta}(r,\sigma)} \mathit{Cl}_\lambda(s) \right) && \text{(by the definition of } \widehat{\delta}^\star\text{)} \\
&= \bigcup_{r \in \{\widehat{q}\}} \left(\bigcup_{s \in \widehat{\delta}(r,\sigma)} \mathit{Cl}_\lambda(s) \right) && \text{(since } \widehat{\delta}^\star(q, \mu) = \{\widehat{q}\}\text{)} \\
&= \bigcup_{s \in \widehat{\delta}(\widehat{q},\sigma)} \mathit{Cl}_\lambda(s) \\
&= \bigcup_{s \in \{\delta(\widehat{q},\sigma)\}} \mathit{Cl}_\lambda(s) && \text{(since } \widehat{\delta}(\widehat{q}, \sigma) = \{\delta(\widehat{q}, \sigma)\}\text{)} \\
&= \mathit{Cl}_\lambda(\delta(\widehat{q}, \sigma)) \\
&= \{\delta(\widehat{q}, \sigma)\} && \text{(by Lemma 1)} \\
&= \{\delta^\star(q, \nu)\} && \text{(as noted above).}
\end{aligned}$$

Now, since ν was an arbitrarily chosen string from $\Sigma^\star$ such that $|\nu| = k+1$, it follows that $\widehat{\delta}^\star(q, \nu) = \{\delta^\star(q, \nu)\}$ for every state $q \in Q$ and for *every* string $\nu \in \Sigma^\star$ with length $k+1$ — as required to establish the Inductive Claim.

This completes the Inductive Step. The claim now follows by induction on the length of ω. □

Lemma 3. *If M and $\widehat{M}$ are as above then $L(\widehat{M}) = L(M)$.*

Proof. Let $\omega \in \Sigma^\star$ such that $\omega \in L(M)$. Then it follows (by the definition of $L(M)$) that $\delta^\star(q_0, \omega)$ is an accepting state in M — that is, $\delta^\star(q_0, \omega) \in F$. Now $\widehat{\delta}^\star(q_0, \omega) = \{\delta^\star(q_0, \omega)\}$

by Lemma 2, above, so that $\widehat{\delta}^\star(q_0, \omega) \cap F = \{\delta^\star(q_0, \omega)\} \neq \emptyset$, and (by the definition of the acceptance of a string by a nondeterministic finite automaton) $\widehat{M}$ accepts ω. That is, $\omega \in L(\widehat{M})$. Since ω was arbitrarily chosen from $L(M)$ it follows that $L(M) \subseteq L(\widehat{M})$.

Now let $\omega \in \Sigma^\star$ such that $\omega \in L(\widehat{M})$. It follows (by the definition of $L(\widehat{M})$) that $\widehat{\delta}^\star(q_0, \omega) \cap F \neq \emptyset$. Now $\widehat{\delta}^\star(q_0, \omega) = \{\delta^\star(q_0, \omega)\}$ by Lemma 2, above, so that $\delta^\star(q_0, \omega)$ must belong to F. That is, $\omega \in L(M)$. Since ω was arbitrarily chosen from $L(\widehat{M})$ it follows that $L(\widehat{M}) \subseteq L(M)$ as well.

Thus $L(\widehat{M}) = L(M)$, as claimed. □

Corollary 4. *Let $L \subseteq \Sigma^\star$ for an alphabet Σ. If L is a regular language then L is the language of a nondeterministic finite automaton.*

Proof. If L is a regular language then $L = L(M)$ for some deterministic finite automaton M. It follows by Lemma 3 that $L = L(\widehat{M})$ as well, where $\widehat{M}$ is the nondeterministic finite automaton obtained by M by applying the above construction. Thus L is the language of a nondeterministic finite automaton as well and, since L was an arbitrarily chosen regular language, this establishes the claim. □

Simulating a Nondeterministic Finite Automaton

Once again, let Σ be an alphabet — but now let

$$M = (Q, \Sigma, \delta, q_0, F)$$

be a ***nondeterministic finite automaton*** with this alphabet, and let $L \subseteq \Sigma^\star$ be the language of M. Since M is a *nondeterministic* finite automaton, δ is now a total function with domain $Q \times \Sigma_\lambda$ and range $\mathcal{P}(Q)$. Our goal, for this part of the proof, is to describe a *deterministic* finite automaton

$$\widehat{M} = (\widehat{Q}, \Sigma, \widehat{\delta}, \widehat{q_0}, \widehat{F})$$

with the same alphabet (but not, generally, the same set of states) such that $L = L(\widehat{M})$ as well.

Recall that a *construction* of $\widehat{M}$ from M was given in the lecture notes, and that this included the definition of a total function

$$\varphi : Q \to \mathcal{P}(Q).$$

It will turn out that φ is an ***injective function*** — that, is, if $r, s \in Q$ such that $r \neq s$ then $\varphi(r) \neq \varphi(s)$ as well. Now $\varphi(q)$ will be the subset of Q that "corresponds to" any state $q \in \widehat{Q}$, as this was suggested in the lecture notes.

As described in the lecture notes, we will name the states of $\widehat{Q}$ so that, at the end of the construction,

$$\widehat{Q} = \{\widehat{q}_0, \widehat{q}_1, \dots, \widehat{q}_{m-1}\}$$

where m is the number of states that we need to include.

Giving the Construction as an Algorithm

As described in the lecture notes, a construction can be used to discover the set $\widehat{Q}$ (by determining which subsets of Q are needed) — determining the transition function $\widehat{\delta}$, for $\widehat{M}$, along the way. A version of this construction is given in Figure 1 on page 6.

This is the same — nonessential — change will be made to simplify the description. Let $m = |\Sigma|$ and, renaming symbols if necessary, let us assume that

$$\Sigma = \{\sigma_0, \sigma_1, \dots, \sigma_{m-1}\}.$$

It will be assumed that symbols in Σ are considered by increasing order of their index when the construction is used.[1]

The final "`return`" statement from the version in the lecture notes is also not really needed, if we think of $\widehat{Q}$ and $\widehat{\delta}$ as global data that are being modified, so it has also been deleted from this version of the algorithm.

If you compare this to the version in the lecture notes then you should be able to confirm that the only changes that have been made, here, the the ones needed to reflect the changes that have now been described.

Correctness of the Construction

Proving the "correctness' of the algorithm shown in Figure 1 involves the use of concepts, and techniques, from other courses. Students who are not already familiar with this material or not interested in a proof of this construction's correctness (or both) can skip the rest of this document.

With that noted, consider the "Key Properties" given in Figure 2 on page 7. Consider, as well, the additional properties given in Figure 3 on page 7.

Lemma 5. *Consider an execution of the body of the inner loop of the construction (at lines* 9*–*14*). If the properties given in Figure 3 are satisfied at the beginning of this execution of the inner loop body, then this execution ends after a finite number of steps have been taken, and*

[1] A version of the construction that did not include this assumption could also be stated and shown and proved to be correct, but various statements about the algorithm would be more complicated, and potentially more confusing, without this assumption.

1. Set $\widehat{q}_0$ to be a state such that $\varphi(\widehat{q}_0) = \mathit{Cl}_\lambda(q_0)$ — setting $\widehat{\delta}(\widehat{q}_0, \sigma)$ to be undefined, for every symbol $\sigma \in \Sigma$.
2. `integer i` := 1
3. $\widehat{R} := \{\widehat{q}_0\}$
4. $\widehat{Q} := \emptyset$
5. `while` $(\widehat{R} \neq \emptyset)$ `{`
6.   Choose a state $\widehat{q}$ from $\widehat{R}$; let V be the set of states, in the given NFA, such that $\varphi(\widehat{q}) = V$
7.   `integer j` := 0
8.   `while` $(\mathtt{j} < |\Sigma|)$ `{`
9. $W := \bigcup_{r \in V} \left(\bigcup_{s \in \delta(r, \sigma_{\mathtt{j}})} \mathit{Cl}_\lambda(s) \right)$
10.   Set $\widetilde{q}$ to be a (possibly new) state such that $\varphi(\widetilde{q}) = W$ — reusing an existing state, if one already exists; $\widehat{\delta}(\widehat{q}, \sigma_{\mathtt{j}})$ will be set to be $\widetilde{q}$
11.   `if` ($\widetilde{q}$ is a new state) `{`   // Update `i` and $\widehat{R}$
12. $\widehat{q}_{\mathtt{i}} := \widetilde{q}$
13. $\mathtt{i} := \mathtt{i} + 1$
14. $\widehat{R} := \widehat{R} \cup \{\widetilde{q}\}$
        `}`
15. $\mathtt{j} := \mathtt{j} + 1$
      `}`
16. $\widehat{R} := \widehat{R} \setminus \{\widehat{q}\}$
17. $\widehat{Q} := \widehat{Q} \cup \{\widehat{q}\}$
    `}`

Figure 1: Construction to Compute the Set $\widehat{Q}$ of States and Transition Function $\widehat{\delta} : \widehat{Q} \times \Sigma \to \widehat{Q}$

the properties in Figure 3 are satisfied, once again, when this execution of the inner loop body ends.

Proof. Consider an execution of the inner loop body such that the properties in Figure 3 are initially satisfied. Since property #3 is satisfied, j is an integer such that $0 \leq \mathtt{j} \leq |\Sigma|$ — but, furthermore, the test at line 8 must have been checked and passed, $j \leq |\Sigma| - 1$, so that $\sigma_{\mathtt{j}}$ is a symbol in Σ and one can see, by an examination of the set at line 9, that (after its execution)

Key Properties:

1. $\mathtt{i}$ is an integer variable whose value is positive.
2. $\widehat{R}$ and $\widehat{Q}$ are finite sets such that $\widehat{Q} \cup \widehat{R} = \{\widehat{q}_0, \widehat{q}_1, \ldots, \widehat{q}_{\mathtt{i}-1}\}$ and $\widehat{Q} \cap \widehat{R} = \emptyset$.
3. φ is a total function from $\widehat{Q} \cup \widehat{R}$ to $\mathcal{P}(Q)$ such that if $x, y \in \widehat{Q} \cup \widehat{R}$ and $x \neq y$ then $\varphi(x) \neq \varphi(y)$.
4. $\widehat{\delta}$ is a partial function from $(\widehat{Q} \cup \widehat{R}) \times \Sigma$ to $(\widehat{Q} \cup \widehat{R})$ such that $\widehat{\delta}(x, \sigma)$ is defined for every state $x \in \widehat{Q}$ and for every symbol $\sigma \in \Sigma$.
5. For all states $x, y \in \widehat{Q} \cup \widehat{R}$, and for every symbol $\sigma \in \Sigma$, if $\widehat{\delta}(x, \sigma) = y$, then

$$\varphi(y) = \bigcup_{r \in \varphi(x)} \left(\bigcup_{s \in \delta(r,\sigma)} \mathit{Cl}_\lambda(s) \right). \tag{3}$$

Figure 2: “Key Properties” Maintained During an Application of the Construction

Loop Invariant:

1. The “Key Properties”, given in Figure 2, are satisfied.
2. $\widehat{q}$ is a state in $\widehat{R}$, and $V = \varphi(\widehat{q}\,) \subseteq Q$.
3. $\mathtt{j}$ is an integer such that $0 \leq \mathtt{j} \leq |\Sigma|$.
4. $\widehat{\delta}(\widehat{q}, \sigma_\ell)$ is defined, for every integer ℓ such that $0 \leq \ell \leq \mathtt{j} - 1$.

Figure 3: Loop Invariant for the Inner Loop in the Construction

W is a subset of Q such that

$$W = \bigcup_{r \in V} \left(\bigcup_{s \in \delta(r,\sigma_{\mathtt{j}})} \mathit{Cl}_\lambda(s) \right) = \bigcup_{r \in \varphi(q)} \left(\bigcup_{s \in \delta(r,\sigma_{\mathtt{j}})} \mathit{Cl}_\lambda(s) \right)$$

— since $V = \varphi(q)$ for the state $q \in \widehat{R}$, since property #2 is initially satisfied too.

The steps at lines 10 and 11 are reached and executed; either the test at line 11 is passed when checked, or it is not. These cases are considered separately, below.

- Suppose, first, that the test at line 11 is passed when it is checked. Then (at this point) there is no state $t \in \widehat{Q} \cup \widehat{R}$ such that $\varphi(t) = W$. In this case, the execution of the loop body will end after the execution of the steps at 12–15 (only a finite number of steps, as claimed).

Consider the Key Properties shown in Figure 2.

- Since part #1 of the Key Properties were initially satisfied, $\mathtt{i}$ is a positive integer when this execution of the inner loop body began. Since the only step that changes the value of $\mathtt{i}$ is the step at line 13, and this increases the value of $\mathtt{i}$ by one, $\mathtt{i}$ is still a positive integer, and part #1 of the Key Properties is still is satisfied, when this execution of the inner loop body ends.
- Part #2 of the Key Properties is initially satisfied, so that $\widehat{Q}$ and $\widehat{R}$ are finite sets satisfied the conditions listed there. After the execution of the steps in the inner loop body, the value of $\mathtt{i}$ has been increased by one — but a new state, called "$\widehat{q}_{\mathtt{i}}$" before the value of $\mathtt{i}$ has been increased, but called "$\widehat{q}_{\mathtt{i}-1}$" after the value of $\mathtt{i}$ has been adjusted, has been introduced and added to $\widehat{Q} \cup \widehat{R}$. Thus is is true (once again) that

  $$\widehat{Q} \cup \widehat{R} = \{\widehat{q}_0, \widehat{q}_1, \dots, \widehat{q}_{\mathtt{i}-1}\}$$

  when this execution of the inner loop body ends. Furthermore, since this new state was not also added to $\widehat{Q}$, it is still true that $\widehat{Q} \cap \widehat{R} = \emptyset$ when this execution of the inner loop body ends. Thus part #2 of the Key Properties is also satisfied at this point.
- Part #3 of the Key Properties is initially satisfied, so that φ is a total function from $\widehat{Q} \cup \widehat{R}$ to $\mathcal{P}(Q)$ such that if $x, y \in \widehat{Q} \cup \widehat{R}$ and $x \neq y$, the $\varphi(x) \neq \varphi(y)$. Now, when the steps in the inner loop body are executed, one more state — called $\widehat{q}_{\mathtt{i}-1}$ at end of the execution of these steps — has been included in $\widehat{Q} \cup \widehat{R}$, and $\varphi(\widehat{q}_{\mathtt{i}-1})$ has been set to be W. Now, $\varphi(x)$ has not been changed for any state x that already belonged to $\widehat{Q} \cup \widehat{R}$, and the new state has only been introduced after it has been confirmed that $\varphi(x) \neq W$, for all the states x that already belonged to $\widehat{Q} \cup \widehat{R}$. Thus $\varphi(x) \neq \varphi(\widehat{q}_{\mathtt{i}})$ for every state x that belonged to $\widehat{Q} \cup \widehat{R}$ when the execution of the loop body began. Furthermore, is it also true that $\varphi(x) \neq \varphi(y)$ for all states $x, y \in \widehat{Q} \cup \widehat{R}$ such that $x \neq y$ at the end of this execution of the loop body — and part #3 is also satisfied, once again, at this point.
- While the partial function $\widehat{\delta}$ is changed during an execution of the inner loop body, it is only changed because the value $\widehat{\delta}(x, \sigma)$ is set, for one more state $x \in \widehat{R}$ and symbol $\sigma \in \Sigma$. Since the set $\widehat{Q}$ is not changed, this partial function is still defined at (x, σ) for every state $x \in \widehat{Q}$ and for every symbol $\sigma \in \Sigma$, and part #4 is still satisfied, at the end of this execution of the inner loop body (because it was satisfied when this execution began).
- While the set $\widehat{R}$ is extended, by including the new state that is (ultimately) called "$\widehat{q}_{\mathtt{i}-1}$", the value of the partial function $\widehat{\delta}$ is not changed for any state and symbol

where it was already defined (and the value of φ is never changed when it was already defined, either). Thus — in order to check that part #5 of the “Key Properties” is satisfied at the end of the execution, if was satisfied initially — the only thing that remains to be checked is that

$$\varphi(\widehat{q}_{\mathtt{i}-1}) = \bigcup_{r \in \varphi(\widehat{q})} \left(\bigcup_{s \in \delta(r, \sigma_{\mathtt{j}})} \mathit{Cl}_\lambda(s) \right)$$

when this execution of the inner loop body ends. Since $\varphi(\widehat{q}_{\mathtt{i}-1})$ is set to be W (as a result of the execution of the step at line 12) and $V = \varphi(\widehat{q})$ (by property #2, as given in Figure 3), this follows, once again, by the definition of W at line 9 in the algorithm. Thus part #5 of the “Key Properties” is satisfied at the end of this execution of the body of the inner loop.

Thus the “Key Properties” are satisfied — as is property #1 in Figure 3 — at the end of this execution of the inner loop body.

Since property #2 in Figure 3 was satisfied at the beginning of this execution of the inner loop body, $\widehat{q}$ is a state in $\widehat{R}$ and $V = \varphi(\widehat{q}) \subseteq Q$ this point. One can see by inspection of the loop body (that is, the steps at lines 9–15) that $\widehat{q}$, V, and $\varphi(\widehat{q})$ are not changed by an execution of the inner loop body. While the set $\widehat{R}$ might be changed (if the step at line 16 is reached), no elements are removed from this set. Thus it is still the case that $\widehat{q}$ is a state in $\widehat{R}$ and $V = \varphi(\widehat{q}) \subseteq Q$ when this execution of the inner loop body ends. That is, property #2 in Figure 3 is also satisfied at this point.

Since property #3 in Figure 3 is satisfied at the beginning of this execution of the inner loop body, $\mathtt{j}$ is an integer such that $0 \le \mathtt{j} \le |\Sigma|$. Furthermore, the test at line 8 must have been checked and passed at this point, $\mathtt{j} < |\Sigma|$ and, since $\mathtt{j}$ and $|\Sigma|$ are both integers, $0 \le \mathtt{j} \le |\Sigma| - 1$. Now, while $|\Sigma|$ is not changed by the execution of the inner loop body, the value of $\mathtt{j}$ is increased by one when the step at line 15 is executed. Thus $1 \le \mathtt{j} \le |\Sigma|$ when the execution of the loop body ends — and property #3 in Figure 3 is still satisfied at this point.

Since property #4 in Figure 3 is satisfied at the beginning of this execution of the loop body, $\widehat{\delta}(\widehat{q}, \sigma_\ell)$ is defined for every integer ℓ such that $0 \le \ell \le \mathtt{j}-1$. Since the test at line 8 is passed, the steps at line 9 and 10 are reached and executed and, after this, $\widehat{\delta}(\widehat{q}, \sigma_\ell)$ is defined for every integer ℓ such that $0 \le \ell \le \mathtt{j}$. Once the step at line 15 is reached, the value of $\mathtt{j}$ is increased by one — so that $\widehat{\delta}(\widehat{q}, \sigma_\ell)$ is defined for every integer ℓ such that $0 \le \ell \le \mathtt{j} - 1$, and property #4 is satisfied, at the end of this execution of the inner loop body, once again.

Thus the properties in Figure 3 are all satisfied, at the end of this execution of the loop body, in this case.

- Suppose, instead, that the test at line 11 is failed when it is checked. Then there exists a state $\widetilde{q} \in \widehat{Q} \cup \widehat{R}$ such that $\varphi(\widetilde{q}\,) = W$. In this case, this execution of the loop body will end after the execution of the step at line 15 (only a finite number of steps, once again).

  Consider the Key Properties shown in Figure 2.

  – Since the integer `i`, sets $\widehat{Q}$ and $\widehat{R}$, and function φ are not changed by any of the steps that are executed, parts #1, #2 and #3 of these Key Properties are satisfied when this execution of the inner loop body ends, because they are unaffected by its execution and were satisfied when this execution of the inner loop body started.

  – The reason why part #4 of the Key Properties is satisfied, at the end of this execution of the inner loop body, is essentially the same as it is for the first case, as given above.

  – Once again, the partial function $\widehat{\delta}$ is changed in only one way during the execution of the inner loop body: $\widehat{\delta}(\widehat{q}, \sigma_{\mathtt{j}})$ is set to be $\widetilde{q}$ at line 10. Thus the only thing that must be checked, in order to confirm that part #5 is satisfied at the end of execution of the inner loop body (if it was satisfied at the beginning) is that

  $$\varphi(\widetilde{q}\,) = \bigcup_{r \in \varphi(\widehat{q}\,)} \left(\bigcup_{s \in \delta(r, \sigma_{\mathtt{j}})} \mathit{Cl}_{\lambda}(s) \right).$$

  Since $\varphi(\widetilde{q}\,) = W$ (by the choice of $\widetilde{q}$ at line 10) and $V = \varphi(\widehat{q}\,)$ (by property #2, as given in Figure 3), this follows by the definition of W at line 9 in the algorithm. Thus part #5 of the "Key Properties" is satisfied at the end of this execution of the body of the inner loop.

  Thus the "Key Properties" are satisfied — as is property #1 in Figure 3 — at the end of this execution of the inner loop body.

  Properties #2, #3, and #4 in Figure 3 are satisfied at the end of this execution of the inner loop body for the same reasons as given in the first case, above.

  Thus the properties in Figure 3 are all satisfied, at the end of this execution of the loop body, in this case, as well.

Thus all properties listed in Figure 3 are satisfied, at the end of this execution of the inner loop body, in every possible case — as needed to establish the claim. □

Lemma 6. *Consider an execution of the body of the outer loop (lines 6–17) that begins with the properties in Figure 2 satisfied. Then the properties given in Figure 3 are satisfied when the inner loop is reached. Furthermore, if ℓ is an integer such that the body of the inner loop is executed at least ℓ times, during this execution of the body of the outer loop, then the properties in Figure 3 are also satisfied immediately after the ℓ^{th} execution of the body of the inner loop.*

Proof. Note, first, that the properties in Figure 2 depend only on the integer `i`, the sets $\widehat{Q}$ and $\widehat{R}$, and the functions φ and $\widehat{\delta}$. None of these values are changed by the execution of the steps at line 6 or 7, so these properties are still satisfied when the inner loop at lines 8–15 is reached. These values are also unchanged by an execution of the loop test at line 8, so the properties in Figure 2 are also still satisfied at the beginning of the first execution of the body of the inner loop. That is, property #1 in Figure 3 is satisfied at this point.

The loop test (for the outer loop) at line 5 must have been checked and passed in order for the body of the outer loop to be executed at all, so that $\widehat{R} \neq \emptyset$ when the step at line 6 is reached. It is therefore possible to choose a state $\widehat{q} \in \widehat{R}$ as described at line 6. Furthermore in follows by property #3 in Figure 2 that $\varphi(\widehat{q})$ is defined (and $\varphi(\widehat{q}) \subseteq Q$), so that V can be set to be $\varphi(\widehat{q}) \subseteq Q$, as described at line 6, as well. Since neither $\widehat{q}$ nor V are changed by either the execution of the step at line 7 or the loop test at line 8, property #2 in Figure 3 is also satisfied, both when the inner loop is first reached, and at the beginning of the first execution of the body of this loop.

The integer variable `j` is defined, and initialized with value 0, when the step at line 7 is executed. Since the value of this variable is not changed when the inner loop test at line 8 is checked, property #3 in Figure 3 is also satisfied, both when the inner loop is first reached, and at the beginning of the first execution of the body of this loop.

Note that property #4 in Figure 3 is "vacuous" (or "empty") — it does not assert anything at all, so that it is trivially satisfied — whenever $\mathtt{j} = 0$. Since $\mathtt{j} = 0$ when the inner loop is first reached, and the beginning of the first execution of the body of this loop.

Thus all properties in Figure 3 are satisfied both when the inner loop is first reached, and at the beginning of the first execution of the body of this loop — as needed to establish the beginning of the claim. The rest of the claim can now be established by the mathematical induction on ℓ, using the standard form of mathematical induction.

Basis: Let $\ell = 1$ (noting that, since `j` has initial value 0, and $|\Sigma| \geq 1$, the loop test is initially passed, so that there *is* at least one execution of the body of the inner loop). As noted above, all properties in Figure 3 are satisfied at the *beginning* of the first (that is, ℓ^{th}) execution of the body of the inner loop. It now follows by Lemma 5 that this first execution of the loop body ends after a finite number of steps, and the properties in Figure 3 are satisfied when this execution of the inner loop body ends. Thus the second part of the claim is satisfied when $\ell = 1$.

Inductive Step: Let k be an integer such that $k \geq 1$. It is necessary and sufficient to use the following

> Inductive Hypothesis: Consider an execution of the body of the outer loop that begins with the properties in Figure 2 satisfied. If the body of the inner loop is executed at least k times (during this execution of the outer loop body), then the properties in Figure 3 are satisfied at the end of the k^{th} execution of the inner loop body.

to prove the following

> Inductive Claim Consider an execution of the body of the outer loop that begins with the properties in Figure 2 satisfied. If the body of the inner loop is executed at least $k+1$ times (during this execution of the inner loop body), then the properties in Figure 3 are satisfied at the end of the $k+1^{\text{st}}$ execution of the inner loop body.

With that noted, suppose that the inner loop body is executed at least $k+1$ times during this execution of the outer loop body (because the Inductive Claim is trivially satisfied, otherwise). Then the inner loop body is certainly executed at least k times and it follows by the Inductive Hypothesis that the properties in Figure 3 are all satisfied at the end of the k^{th} execution of the body of the inner loop.

As noted above, none of these properties are affected by an execution of the inner loop test, so the properties in Figure 3 are also satisfied at the beginning of the $k+1^{\text{st}}$ execution of the inner loop body. Once again, it follows by Lemma 5 that these properties are all satisfied at the end of the $k+1^{\text{st}}$ execution of the inner loop body, as well.

Thus the Inductive Claim is satisfied — as needed to complete both the Inductive Step, and this proof of the claim — because the second part of the claim now follows by induction on ℓ. □

Lemma 7. *Consider an execution of the body of the outer loop, at lines* 6–17*. If the properties in Figure 2 are satisfied at the beginning of the execution of the loop body, and this execution of the loop body eventually ends, then the properties in Figure 2 are satisfied at the end of this execution of the loop body, as well.*

Proof. Consider an execution of the body of the outer loop that begins with the properties in Figure 2 satisfied. Then, after the steps at lines #6 and #7, the inner loop at lines #8–#15 is reached and executed. It follows by Lemma 6, above, that the properties in Figure 3 (which include the properties in Figure 2) are satisfied when the inner loop is first reached and, also, at the end of every execution of the body of this inner loop.

Either the execution of the inner loop eventually ends, or it does not.

- If the execution of the inner loop eventually ends then — since the final execution of the loop test does not the change the values of any variables, sets or functions, the properties in Figure 3 are still all satisfied when the execution of the inner loop ends (so that the step at line #14 is about to be executed).
  - Property #1 in Figure 2 is still satisfied, at the end of this execution of the body of the outer loop, because an executions of the steps at line #16 and #17 do not change the value (or type) of the integer variable `i`.
  - The execution of the steps at line #16 and #17 change the sets $\widehat{Q}$ and $\widehat{R}$. However, it does so by removing one element, $\widehat{q}$, from $\widehat{R}$ and including it in $\widehat{Q}$. As a result,

$\widehat{Q}$ and $\widehat{R}$ are still finite sets (of subsets of Q) such that $\widehat{Q} \cap \widehat{R} = \emptyset$, and such that $\widehat{Q} \cup \widehat{R}$ has not been changed by the execution of this step — so that

$$\widehat{Q} \cup \widehat{R} = \{\widehat{q}_0, \widehat{q}_1, \ldots, \widehat{q}_{\mathtt{i}-1}\}$$

once again — and so that Property #2 in Figure 2 is satisfied at the end of this execution of the body of the outer loop, as well.

– Since Property #3 in Figure 2 is satisfied immediately before the execution of the steps at line #16 and #17, and since the execution of these steps does not change either the partial function φ or the set $\widehat{Q} \cup \widehat{R}$, this property is satisfied after the execution of this step (that is, at the end of this execution of the body of the loop), as well.

– Since property #3 in Figure 3 is satisfied immediately before the execution of the step at line #16, $\mathtt{j}$ is an integer variable such that $0 \leq \mathtt{j} \leq |\Sigma|$ at this point. Furthermore, the inner loop test at line #8 must have been checked and failed, so that $\mathtt{j} \not< |\Sigma|$. Thus $\mathtt{j} = |\Sigma|$ at this point — and, since Property #4 in Figure 3 is also satisfied, $\widehat{\delta}(\widehat{q}, \sigma)$ is defined for every symbol $\sigma \in \Sigma$.

  Now, since Property #4 in Figure 2 is satisfied immediately before the execution of the step at line #16, φ is a partial function from $(\widehat{Q} \cup \widehat{R}) \times \Sigma$ to $(\widehat{Q} \cup \widehat{R})$ such that $\widehat{\delta}(x, \sigma)$ is defined for every state $x \in \widehat{Q} \cup \{\widehat{q}\}$, and for every symbol $\sigma \in \Sigma$, before the execution of the step at line #16. Since $\widehat{Q}$ is only changed by the inclusion of $\widehat{q}$ (and φ and $\widehat{Q} \cup \widehat{R}$ are not changed at all) when the steps at lines #16 and #17 is executed, it follows that $\widehat{\delta}(x, \sigma)$ is defined for every state $x \in \widehat{Q}$ and every symbol $\sigma \in \Sigma$, and Property #4 in Figure 2 is satisfied, at the end of this execution of the body of the loop.

– As for Property #3, Property #5 is also satisfied after the execution of the step at lines #16 and #17 (so that it is satisfied at the end of this execution of the body of the loop) because it is satisfied immediately before these steps are executed, and it only concerns the partial function φ and the set $\widehat{Q} \cup \widehat{R}$ — which are not changed by the execution of this step.

Thus the properties in Figure 2 are satisfied, at the end of this execution of the body of the outer loop — and the claim holds in this case.

- If the execution of the inner loop never ends then this execution of the body of the outer loop certainly never ends either and the claim is trivially satisfied (because there is nothing that must be proved in this case).

Thus the claim holds in every possible case, as required. □

It is necessary to eliminate the possibility that the execution of the body of the loop never ends, and the following result is needed to establish this.

Lemma 8. *Consider an execution of the inner loop, that is part of an execution of the body of the outer loop, beginning with the properties in Figure 2 satisfied. If $h = |\Sigma| - j$ when the loop test at line 6 is executed then h is an integer such that $0 \leq h \leq |\Sigma|$, and the body of the inner loop will be executed exactly h times before the execution of the inner loop ends.*

Proof. Consider an execution of the inner loop, that is part of an execution of the body of the outer loop, beginning with the properties in Figure 2 satisfied. It follows by Lemma 6 that the properties given in Figure 3 are satisfied when the inner loop is first reached, and at the beginning of every execution of the inner loop body. Since an execution of the inner loop test (at line #8) does not change the values of any variables, or have any other effects, these properties are still satisfied after this test is executed — so that they are satisfied at the beginning of every execution of the body of the inner loop as well. It now follows, by Property #3 in Figure 3, that $\mathtt{j}$ is an integer such that $0 \leq \mathtt{j} \leq |\Sigma|$. Since $h = |\Sigma| - \mathtt{j}$, h is an integer such that $0 \leq h \leq |\Sigma|$, at this point, as well.

The rest of the claim can now be established by induction on h: For the basis, one should use the fact that if $h = 0$ then $\mathtt{j} = |\Sigma|$, so that the loop test fails and there are no further executions of the body of the loop. For the Inductive Step, one should note that every execution of the body of the inner loop includes an execution of the step at line #15, which increases the value of $\mathtt{j}$ by one. No other steps change the value of either $\mathtt{j}$ or $|\Sigma|$, so that every execution of the body of the loop *decreases* the value of h by exactly one. This allows the Inductive Hypothesis to be used to establish the Inductive Claim.

The completion of this proof (by following the above outline to write out the remaining details) is left as an ***exercise***. $\square$

Lemma 9. *Consider an execution of the body of the outer loop, at lines 6–17. If the properties in Figure 2 are satisfied at the beginning of the execution of the loop body, then this execution of the loop body eventually ends (after the execution of a finite number of steps).*

Proof. Consider an execution of the body of the outer loop if the properties in Figure 2 are initially satisfied. The test for the *inner* loop (at line #8) is part of a single step, and the body of the inner loop (at lines #9– #15) includes only a finite number of steps. None of these include other loops or calls to other methods that might fail to halt — so every execution of either the inner loop test, or the body of the inner loop, ends after the execution of a finite number of steps. Now, the variable $\mathtt{j}$ has value 0 when the inner loop is reached, so that $|\Sigma| - \mathtt{j} = |\Sigma|$ at this point, and it follows by Lemma 8 that the execution of the inner loop ends — after the body of the inner loop has been executed $|\Sigma|$ times, and after $|\Sigma| + 1$ executions of the loop test.

The execution of the body of the outer loop includes the execution of only four more steps (at lines #6, #7, #16 and #17), so the execution of the body of the outer loop also ends after the execution of a finite number of steps, as claimed. $\square$

Lemmas 7 and 9 imply the following.

Corollary 10. *If the body of the outer loop (at lines #6–#17) is executed when the properties in Figure 2 are initially satisfied, then the execution of the loop body ends after the execution of a finite number of steps, with the properties in Figure 2 satisfied, once again.*

Lemma 11. *Consider an execution of the algorithm in Figure 1, when $M = (Q, \Sigma, \delta, q_0, F)$ is a nondeterministic finite automaton. Then the "Key Properties" in Figure 2 are satisfied when the outer loop (at lines #5–#17) is first reached.*

Proof. Consider an execution of the algorithm in Figure 1 when $M = (Q, \Sigma, \delta, q_0, F)$ is a nondeterministic finite automaton. The outer loop is reached after the steps at lines #1–#4 are executed.

- At this point, part #1 of the "Key Properties" shown in Figure 2 is satisfied because the variable $\mathtt{i}$ has been declared to be an integer variable at line 2, with (positive) initial value 1.

- The sets $\widehat{R}$ and $\widehat{Q}$ have been initialized, at lines #3 and #4, to be the sets $\{\widehat{q}_0\}$ and $\emptyset$, respectively. Thus (since $\mathtt{i} = 1$ at this point)

  $$\widehat{Q} \cup \widehat{R} = \{\widehat{q}_0\} = \{\widehat{q}_0, \dots, \widehat{q}_{\mathtt{i}-1}\} \qquad \text{and} \qquad \widehat{Q} \cap \widehat{R} = \emptyset$$

  when the loop is first reached, so that part #2 of the "Key Properties" is also satisfied at this point.

- Part #3 of the "Key Properties" is satisfied at this point because $\widehat{Q} \cup \widehat{R} = \{\widehat{q}_0\}$ when the loop is reached, and $\varphi(\widehat{q}_0)$ has been defined to be the subset $\mathit{Cl}_\lambda(q_0)$ of Q at this point — so that φ is a total function from $\widehat{Q} \cup \widehat{R}$ to $\mathcal{P}(Q)$, as this part of the "Key Properties" asserts. Furthermore, since $|\widehat{Q} \cup \widehat{R}| = 1$, no pair of distinct elements x and y of $\widehat{Q} \cup \widehat{R}$ exists at this point, so the second half of this part of the "Key Properties" is trivially satisfied.

- Part #4 of the "Key Properties" is satisfied because $\widehat{\delta}$ has been defined to a partial function from $(\widehat{Q} \cup \widehat{R}) \times \Sigma$ to $\widehat{Q} \cup \widehat{R}$, and $\widehat{Q} = \emptyset$ — so that $\delta(x, \sigma)$ is defined for every element x of $\widehat{Q}$ and every symbol $\sigma \in \Sigma$, as claimed.

- Finally, part #5 of the "Key Properties" is trivially satisfied when the loop is reached because there is no state $x \in \widehat{Q} \cup \widehat{R}$ and symbol $\sigma \in \Sigma$ such that $\delta(x, \sigma)$ is defined at this point

Thus the "Key Properties" are satisfied when the loop is reached, as claimed. □

Lemma 12. *Consider an execution of the algorithm in Figure 1 on input M, where $M = (Q, \Sigma, \delta, q_0, F)$ is a nondeterministic finite automaton. If ℓ is a positive integer such that the body of the outer loop (at lines #5–#17) is executed at least ℓ times, during this execution of the algorithm, then the "Key Properties" in Figure 2 are satisfied at both the beginning and end of the ℓ^{th} execution of the outer loop body.*

Proof. This can be proved by induction on ℓ, using the standard form of mathematical induction.

Basis: Let $\ell = 1$, so that the ℓ^{th} execution of the outer loop body is the first execution of the body of this loop.

It follows by Lemma 11 that the "Key Properties" are satisfied when the outer loop body is reached (during an execution of the algorithm on input M). Since the loop test at line #5 has no side-effects (in particular, it does not change the values of any variables) the "Key Properties" are still satisfied after this test is executed, and the first execution of the body of the outer loop begins. Lemma 7 now implies that the "Key Properties" are satisfied at the end of this first execution of the body of the outer loop, as well.

Inductive Step: Let k be an integer such that $k \geq 1$. It is necessary and sufficient to use the following

> Inductive Hypothesis: If the body of the outer loop is executed at least k times (during an execution of the algorithm on input M) then the "Key Properties" are satisfied at both the beginning and the end of the k^{th} execution of the body of the outer loop.

to prove the following

> Inductive Claim:: If the body of the outer loop is executed at least $k + 1$ times (during an execution of the algorithm on input M) then the "Key Properties" are satisfied at both the beginning and the end of the $k + 1^{\text{st}}$ execution of the body of the outer loop.

With that noted, suppose that the body of the outer loop is executed at least $k+1$ times, during the execution of the algorithm on input M (noting that there is nothing that we need to prove, otherwise). Then the outer loop body is certainly executed at least k times, and it follows by the Inductive Hypothesis that the "Key Properties" are all satisfied at the end of the k^{th} execution of the outer loop body. Since the loop test at line #5 has no side-effects, the "Key Properties" are still satisfied at the beginning of the $k + 1^{\text{st}}$ execution of the outer loop body. Once again, it follows by Lemma 7 that the "Key Properties" are also satisfied at the end of this execution of the loop body, as needed to establish the Inductive Claim and complete the Inductive Step.

The claim now follows by induction on ℓ. □

Lemma 13. *Consider an execution of the algorithm in Figure 1 on input M, where $M = (Q, \Sigma, \delta, q_0, F)$ is a nondeterministic finite automaton. If ℓ is a positive integer such that the body of the outer loop (at lines #6–#17) is executed at least ℓ times, then $|\widehat{Q}| = \ell$ at the end of the ℓ^{th} execution of the body of the outer loop.*

Sketch of Proof. This is easily proved by induction on ℓ — noting that $\widehat{Q} = \emptyset$ (so that $|\widehat{Q}| = 0$) when the first execution of the body of the outer loop begins, and noting that (since $\widehat{R} \neq \emptyset$ and $\widehat{Q} \cap \widehat{R} = \emptyset$ at the beginning of every execution of the loop body) the size of the set $\widehat{Q}$ is increased by one during each execution of the body of the outer loop, when the step at line #17 is executed and something is moved out of $\widehat{R}$ and into $\widehat{Q}$. $\square$

Lemma 14. *If the algorithm in Figure 1 is executed on input M, where $M = (Q, \Sigma, \delta, q_0, F)$ is a nondeterministic finite automaton, then the body of the outer loop (at lines 4–14) is executed at most $2^{|Q|}$ times before the execution of the loop ends.*

Proof. Let ℓ be an integer such that the body of the outer loop is executed at least ℓ times, during the execution of the algorithm on input M. It follows by Lemma 13, above, that $|\widehat{Q}| = \ell$ at the end of the ℓ^{th} of the loop body. Now, since part #3 of the "Key Properties" (in Figure 2) is also satisfied, at this point,

$$\{\varphi(q) \mid q \in \widehat{Q}\} \subseteq \mathcal{P}(Q) \qquad \text{and} \qquad |\{\varphi(q) \mid q \in \widehat{Q}\}| = |\widehat{Q}| = \ell$$

when the ℓ^{th} execution of the loop body ends. Thus

$$\ell = |\{\varphi(q) \mid q \in \widehat{Q}\}| \leq |\mathcal{P}(Q)| = 2^{|Q|},$$

so that the loop body can be executed at most $2^{|Q|}$ times, as claimed. $\square$

Lemma 15. *If the algorithm in Figure 1 is executed on input M, where $M = (Q, \Sigma, \delta, q_0, F)$ is a nondeterministic finite automaton, then this execution of the algorithm eventually ends (after a finite number of steps). On termination*

$$\widehat{Q} = \{\widehat{q}_0, \widehat{q}_1, \widehat{q}_2, \ldots, \widehat{q}_{\ell-1}\}$$

for some integer ℓ such that $1 \leq \ell \leq 2^{|Q|}$ and $\widehat{\delta}$ is a total function from $\widehat{Q} \times \Sigma$ to $\widehat{Q}$.

Proof. Consider an execution of the algorithm in Figure 1 with a nondeterministic finite automaton M (as given above) as input.

It follows by Lemma 12 (which establishes that the "Key Properties" are satisfied when an execution of the loop body begins) and Corollary 10 (which establishes termination, under this condition) that every execution of the body of the outer loop terminates. Since Lemma 14 establishes that the loop body is executed at most $2^{|Q|}$ times, the execution of the outer loop terminates — and, since an execution of the algorithm includes only four additional steps, this execution of the algorithm terminates as well.

It now follows by Lemma 7 that the "Key Properties" in Figure 2 are all satisfied at the end of the final execution of the outer loop body. Since the final execution of the loop test (at line #5) has no side-effects these properties are still satisfied when the execution of the algorithm ends. Thus (by part #2 of these properties)

$$\widehat{Q} \cup \widehat{R} = \{\widehat{q}_0, \widehat{q}_1, \widehat{q}_2, \ldots, \widehat{q}_{\ell-1}\}$$

for some positive integer ℓ (which is the final value of the variable `i`). On the other hand, since the loop test at line #5 failed in order to end the loop's execution, $\widehat{R} = \emptyset$ and

$$\widehat{Q} = \{\widehat{q}_0, \widehat{q}_1, \ldots, \widehat{q}_{\ell-1}\}.$$

$|\widehat{Q}| = \ell \geq 1$ since $\widehat{q}_0 \in \widehat{Q}$ (that is, $\widehat{Q} \neq \emptyset$) and it follows by Lemmas 13 and 14 that $\ell \leq 2^{|Q|}$, as required to establish the first part of the claim.

Since part #4 of the "Key Properties" are satisfied, $\widehat{\delta}$ is a function from $(\widehat{Q} \cup \widehat{R}) \times \Sigma$ to $\widehat{Q} \cup \widehat{R}$ such that $\widehat{\delta}(x, \sigma)$ is defined for every state $x \in \widehat{Q}$ and every symbol $\sigma \in \Sigma$. Once again, $\widehat{R} = \emptyset$ when the algorithm terminates, as noted, above, so it follows that $\widehat{\delta}$ is a total function from $\widehat{Q} \times \Sigma$ to $\widehat{Q}$ — establishing the second part of the claim. □

Now recall that the subset $\widehat{F}$ of $\widehat{Q}$ is defined by including a state $\widehat{q}$ (from $\widehat{Q}$) in $\widehat{F}$ if and only if $\varphi(\widehat{q}\,) \cap F \neq \emptyset$. We have now defined a ***deterministic finite automaton***

$$\widehat{M} = (\widehat{Q}, \Sigma, \widehat{\delta}, \widehat{q}_0, \widehat{F})$$

as desired — and it remains only to show that $L(\widehat{M}) = L(M)$.

Lemma 16. *Consider the extended transition functions*

$$\delta^\star : Q \times \Sigma^\star \to \mathcal{P}(Q) \qquad \textit{and} \qquad \widehat{\delta}^\star : \widehat{Q} \times \Sigma^\star \to \widehat{Q}$$

of the nondeterministic finite automaton M and the deterministic finite automaton $\widehat{M}$, respectively. If $\omega \in \Sigma^\star$ then

$$\varphi(\widehat{\delta}^\star(\widehat{q}_0, \omega)) = \delta^\star(q_0, \omega).$$

Sketch of Proof. This can be proved by induction on $|\omega|$ — using the fact that $\varphi(\widehat{q}_0) = \mathit{Cl}_\lambda(q_0)$ to complete the basis and using part #5 of the "Key Properties" to establish an "Inductive Claim" from an "Inductive Hypothesis", when completing the Inductive Step. □

Theorem 17. *If the nondeterministic finite automaton M and the deterministic finite automaton $\widehat{M}$ are as given above then $L(\widehat{M}\,) = L(M)$.*

Sketch of Proof. This follows immediately from Lemma 16, above, and the definition of $\widehat{F}$ from F. $\square$

Thus $L = L(\widehat{M})$, as required to complete the proof that every language of a nondeterministic finite automaton is also a regular language, as claimed.