CPSC 351 — Tutorial Exercise #1
Additional Practice Problems

About These Problems

These problems will not be discussed during the tutorial, and solutions for these problems will not be made available. They can be used as "practice" problems that can help you practice skills considered in the lecture presentation for Lecture #1, or in Tutorial Exercise #1.

Continuing a Problem from the Tutorial Exercise

Once again, consider the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and let $L_0, L_1, L_2 \subseteq \Sigma^\star$ be languages that satisfy the following properties.

(a) $\lambda \in L_0$, $\lambda \notin L_1$, and $\lambda \notin L_2$.

(b) For every string $\omega \in \Sigma^\star$ such that $\omega \in L_0$, $\omega \cdot \mathtt{a} \in L_0$, $\omega \cdot \mathtt{a} \notin L_1$, and $\omega \cdot \mathtt{a} \notin L_2$.

(c) For every string $\omega \in \Sigma^\star$ such that $\omega \in L_0$, $\omega \cdot \mathtt{b} \notin L_0$, $\omega \cdot \mathtt{b} \in L_1$, and $\omega \cdot \mathtt{b} \notin L_2$.

(d) For every string $\omega \in \Sigma^\star$ such that $\omega \in L_1$, $\omega \cdot \mathtt{a} \notin L_0$, $\omega \cdot \mathtt{a} \notin L_1$, and $\omega \cdot \mathtt{a} \in L_2$.

(e) For every string $\omega \in \Sigma^\star$ such that $\omega \in L_1$, $\omega \cdot \mathtt{b} \notin L_0$, $\omega \cdot \mathtt{b} \in L_1$, and $\omega \cdot \mathtt{b} \notin L_2$.

(f) For every string $\omega \in \Sigma^\star$ such that $\omega \in L_2$, $\omega \cdot \mathtt{a} \notin L_0$, $\omega \cdot \mathtt{a} \notin L_1$, and $\omega \cdot \mathtt{a} \in L_2$.

(g) For every string $\omega \in \Sigma^\star$ such that $\omega \in L_2$, $\omega \cdot \mathtt{b} \notin L_0$, $\omega \cdot \mathtt{b} \notin L_1$, and $\omega \cdot \mathtt{b} \in L_2$.

If you completed Tutorial Exercise #1 then you have proved the following claims.

Claim 1. *Let $\omega \in \Sigma^\star$. Then one of the following cases must hold:*

i. $\omega \in L_0$, $\omega \notin L_1$, *and* $\omega \notin L_2$.

ii. $\omega \notin L_0$, $\omega \in L_1$, *and* $\omega \notin L_2$.

iii. $\omega \notin L_0$, $\omega \notin L_1$, *and* $\omega \in L_2$.

Claim 2. $L_0 = \{\mathtt{a}^n \mid n \in \mathbb{N}\}$

Claim 2 was proved by proving two simpler claims, "Claim 3" and "Claim 4", which (essentially) asserted that $\{\mathtt{a}^n \mid n \in \mathbb{N}\} \subseteq L_0$ and that $L_0 \subseteq \{\mathtt{a}^n \mid n \in \mathbb{N}\}$, respectively. You may use these claims when solving the following problems.

1. Prove the following.

   Claim 5. $L_1 = \{\mathtt{a}^n \cdot \mathtt{b}^m \mid n, m \in \mathbb{N}$ and $m \geq 1\}$.

   You might find that it is helpful to state, and prove, two somewhat simpler claims — such that Claim 5 is a straightforward consequence of them.

2. Prove the following

   Claim 6. $L_2 = \{\omega \in \Sigma^\star \mid \mathtt{ba}$ is a substring of $\omega\}$.

   Once again, you might find that it is helpful to state, and prove, two somewhat simpler claims — such that Claim 6 is a straightforward consequence of them.

Additional Problems

Consider the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ once again. Let $L_3, L_4, L_5 \subseteq \Sigma^\star$ that satisfy the following properties.

(h) $\lambda \in L_3$, $\lambda \notin L_4$, and $\lambda \notin L_5$.

(i) For every string $\omega \in \Sigma^\star$ such that $\omega \in L_3$, $\omega \cdot \mathtt{a} \in L_3$, $\omega \cdot \mathtt{a} \notin L_4$, and $\omega \cdot \mathtt{a} \notin L_5$.

(j) For every string $\omega \in \Sigma^\star$ such that $\omega \in L_3$, $\omega \cdot \mathtt{b} \notin L_3$, $\omega \cdot \mathtt{b} \in L_4$, and $\omega \cdot \mathtt{b} \notin L_5$.

(k) For every string $\omega \in \Sigma^\star$ such that $\omega \in L_4$, $\omega \cdot \mathtt{a} \notin L_3$, $\omega \cdot \mathtt{a} \notin L_4$, and $\omega \cdot \mathtt{a} \in L_5$.

(l) For every string $\omega \in \Sigma^\star$ such that $\omega \in L_4$, $\omega \cdot \mathtt{b} \notin L_3$, $\omega \cdot \mathtt{b} \in L_4$, and $\omega \cdot \mathtt{b} \notin L_5$.

(m) For every string $\omega \in \Sigma^\star$ such that $\omega \in L_5$, $\omega \cdot \mathtt{a} \in L_3$, $\omega \cdot \mathtt{a} \notin L_4$, and $\omega \cdot \mathtt{a} \notin L_5$.

(n) For every string $\omega \in \Sigma^\star$ such that $\omega \in L_5$, $\omega \cdot \mathtt{b} \notin L_3$, $\omega \cdot \mathtt{b} \in L_4$, and $\omega \cdot \mathtt{b} \notin L_5$.

3. Consider the following

   Claim 7. *Let $\omega \in \Sigma^\star$. Then one of the following cases must hold:*

   i. $\omega \in L_3$, $\omega \notin L_4$, *and* $\omega \notin L_5$.
   ii. $\omega \notin L_3$, $\omega \in L_4$, *and* $\omega \notin L_5$.
   iii. $\omega \notin L_3$, $\omega \notin L_4$, *and* $\omega \in L_5$.

   Note the resemblance to Claim 1, above (and consider the possibility that proofs of these claims might be similar). Prove Claim 7.

4. Now consider the following

Claim 8. *Let L_3, L_4, and L_5 be as described above. Then the following conditions hold for every string $\omega \in \Sigma^\star$.*

(a) $\omega \in L_3$ *if and only if* ω *does not end with either "*`b`*" or "*`ba`*".*

(b) $\omega \in L_4$ *if and only if* ω *ends with "*`b`*".*

(c) $\omega \in L_5$ *if and only if* ω *ends with "*`ba`*".*

If you have time, try to prove this in two different ways:

- Try to prove this by mathematical induction, using induction on the length of ω, using the standard form of mathematical induction. The structure of the proof would resemble the proof of "Claim 2" from the lecture presentation for Lecture #1.
- Try to prove this by proving parts (a), (b), and (c) separately — just like Claims 2, 5 and 6, above, were proved.

You should find that it is reasonably easy to prove the claim in one of these ways, and that it is difficult (or impossible) to prove the claim in the other way. Can you see why?