CPSC 351 — Tutorial Exercise #1
Review of Mathematical Foundations
Alphabets, Strings, and Languages

About This Exercise

This exercise is based on the preparatory material for the first lecture — including a review of prerequisite material in discrete mathematics as well as an introduction to alphabets, strings, and languages. It is intended to give students practice in using prerequisite material to prove a result about languages over an alphabet.

If you have time: Please try to solve the problems in this exercise ***before*** attending the tutorial where it will be discussed.

The Problems To Be Solved

Consider the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and let $L_0, L_1, L_2 \subseteq \Sigma^\star$ be languages that satisfy the following properties.

(a) $\lambda \in L_0$, $\lambda \notin L_1$, and $\lambda \notin L_2$.

(b) For every string $\omega \in \Sigma^\star$ such that $\omega \in L_0$, $\omega \cdot \mathtt{a} \in L_0$, $\omega \cdot \mathtt{a} \notin L_1$, and $\omega \cdot \mathtt{a} \notin L_2$.

(c) For every string $\omega \in \Sigma^\star$ such that $\omega \in L_0$, $\omega \cdot \mathtt{b} \notin L_0$, $\omega \cdot \mathtt{b} \in L_1$, and $\omega \cdot \mathtt{b} \notin L_2$.

(d) For every string $\omega \in \Sigma^\star$ such that $\omega \in L_1$, $\omega \cdot \mathtt{a} \notin L_0$, $\omega \cdot \mathtt{a} \notin L_1$, and $\omega \cdot \mathtt{a} \in L_2$.

(e) For every string $\omega \in \Sigma^\star$ such that $\omega \in L_1$, $\omega \cdot \mathtt{b} \notin L_0$, $\omega \cdot \mathtt{b} \in L_1$, and $\omega \cdot \mathtt{b} \notin L_2$.

(f) For every string $\omega \in \Sigma^\star$ such that $\omega \in L_2$, $\omega \cdot \mathtt{a} \notin L_0$, $\omega \cdot \mathtt{a} \notin L_1$, and $\omega \cdot \mathtt{a} \in L_2$.

(g) For every string $\omega \in \Sigma^\star$ such that $\omega \in L_2$, $\omega \cdot \mathtt{b} \notin L_0$, $\omega \cdot \mathtt{b} \notin L_1$, and $\omega \cdot \mathtt{b} \in L_2$.

1. To begin, suppose that we wish to prove that every string in $\Sigma^\star$ belongs to ***exactly one*** of the languages L_0, L_1, or L_2 — so that

$$L_0 \cap L_1 = L_0 \cap L_2 = L_1 \cap L_2 = \emptyset$$

and

$$L_0 \cup L_1 \cup L_2 = \Sigma^\star.$$

Another way to state the desired result is as follows:

Claim 1. *Let $\omega \in \Sigma^\star$. Then one of the following cases must hold:*

i. $\omega \in L_0, \omega \notin L_1,$ *and* $\omega \notin L_2.$

ii. $\omega \notin L_0, \omega \in L_1,$ *and* $\omega \notin L_2.$

iii. $\omega \notin L_0, \omega \notin L_1,$ *and* $\omega \in L_2.$

Suppose, in particular, that we wish to prove this result by ***mathematical induction*** on the length of the string ω — using the *standard* form of mathematical induction.

(a) Write an outline (or "skeleton") for this proof that can be used to complete it. In particular, do the following.

- Write down any material that should appear, in the proof, before the ***basis***.
- Using the material about mathematical induction from the preparatory reading for Lecture #1, as needed, write down the *claim* that you should try to prove in order to *complete* the basis. Note that this is not the same thing as completing the basis!
- Suppose that the ***inductive step*** begins with the following:

  Let k be an integer such that $k \geq 0$.

  Write down the ***inductive hypothesis*** (which says something about strings with length k) and the ***inductive claim*** (which says something about strings with length $k+1$) that you would use in the rest of the inductive step for this proof.
- Write down any material that should appear after the ***inductive step***, in order to complete the proof.

(b) Note that there is only one string — the empty string, λ — that belongs to $\Sigma^\star$, whose is length is zero. Use this to determine the information you would need to give in order to complete the ***basis*** for this proof.

(c) Note that if $\omega \in \Sigma^\star$ and the length of ω is $k+1$ (for $k \geq 0$) then ω is not the empty string, so that either $\omega = \mu \cdot$ a or $\omega = \mu \cdot$ b for some string $\mu \in \Sigma^\star$ with length k. Use this to determine the information that you would need to give, in order to complete the ***inductive step*** for this proof.

(d) Use the information, that you now have, to complete a proof of the above claim. If you have time, try to write the kind of proof that you would submit for assessment (as part of an assignment).

Let Σ be an alphabet. If $\omega \in \Sigma^\star$ and n is a non-negative integer, then ω^n is the concatenation of n copies of the string ω.

For example, if $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, as above, and $\omega = \mathtt{ab} \in \Sigma^\star$, then $\omega^0 = \lambda$ (the empty string), $\omega^1 = \omega = \mathtt{ab}$, $\omega^2 = \mathtt{ab} \cdot \mathtt{ab} = \mathtt{abab}$, and $\omega^3 = \mathtt{ab} \cdot \mathtt{ab} \cdot \mathtt{ab} = \mathtt{ababab}$.

2. Suppose, now, that we wish to prove the following.

   Claim 2. $L_0 = \{\mathtt{a}^n \mid n \in \mathbb{N}\}$

   — where $\mathbb{N}$ denotes the set of *non-negative* integers (so that $0 \in \mathbb{N}$).

   (a) Prove the following.

   Claim 3. *For every integer n such that $n \geq 0$, $\mathtt{a}^n \in L_0$.*

   (b) Next prove the following.

   Claim 4. *For every string $\omega \in \Sigma^\star$, if $\omega \in L_0$ then there exists a non-negative integer n such that $\omega = \mathtt{a}^n$.*

   You may find it to be helpful to use Claim 1 when proving this result.

   Note that Claims 3 and 4 imply Claim 2, so you will also have proved Claim 2 if you completed parts (a) and (b).

   Suggestion: When answering the second question, consider using the process that was followed when answering the first question. That is, choose a proof technique to use for each part of the problem. Use this to write an outline for the proof that you need. Try to figure out what information you need to find, and present, in order to fill in the pieces of the proof that you identified. Then write it all up.

What Will Happen During the Tutorial: If possible, students will form groups based on how far they got. The teaching assistant will visit groups, as time allows, to help move discussions along — noting questions that were hard to answer, in case the instructor can clarify things later.

Solutions for the problems in this exercise will be made available shortly after all tutorials for this exercise have concluded. Please not that you might understand somebody else's solution for a problem, without knowing how to solve a problem yourself: ***Understanding the solution is not as useful as completing the exercise yourself. It might not even be as useful as trying to complete the exercise yourself, getting stuck, and thinking about why that happened.***