THE UNIVERSITY OF CALGARY

FACULTY OF SCIENCE

FINAL EXAMINATION

Computer Science 351 — Practice Examination

DATE: December, 2024 TIME: 2 hrs.

NAME: ____

Academic integrity is the foundation of the development and acquisition of knowledge and is based on values of honesty, trust, responsibility, and respect. We expect members of our community to act with integrity. Research integrity, ethics, and principles of conduct are key to academic integrity. Members of our campus community are required to abide by our institutional code of conduct and promote academic integrity in upholding the University of Calgary's reputation of excellence.

Aids Allowed: One double-sided letter-sized page of notes. No other aids are allowed.

Instructions:

- Answer ALL questions in Part A. You may also answer *at most one* ***bonus question*** in Part B. If you start to answer both and do not ***clearly*** show which one should be marked, then only the first will be read and marked.
- Answer questions in the space provided. Continue answers on the blank pages at the end of the test if you need more room.
- This test is out of 30 (the number of marks available in Part A).

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Part A: Answer ALL Questions

1. Let n be a positive integer. Suppose that two basketball players are taking turns, trying to throw a ball into a basket, with each trying to do this n times. (The "first player" goes first.) Each can either "Hit" (that is, have the basketball go through the "Hoop") or "Miss" each time, so that this can be modelled using a ***sample space***

$$\Omega = \{(\alpha_1, \alpha_2, \dots, \alpha_{2n}) \mid \alpha_i \in \{\mathtt{H}, \mathtt{M}\} \text{ for } 1 \le i \le 2n\}$$

whose size is 2^{2n}.

The first player is better at this than the second: Each time the first player tries to throw the ball into the basket, the ball goes through the ***hoop*** with probability $\frac{3}{4}$ and it ***misses*** with probability $\frac{1}{4}$. On the other hand, each time the second player tries to throw the ball into the basket, the ball goes through the ***hoop*** with probability $\frac{1}{4}$ and it ***misses*** with probability $\frac{3}{4}$.

This can be modelled using a ***probability distribution*** $\mathsf{P} : \Omega \to \mathbb{R}$ such that, if $0 \le k, \ell \le n$, and $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_{2n}) \in \Omega$ such that exactly k of $\alpha_1, \alpha_3, \dots, \alpha_{2n-1}$ are "$\mathtt{H}$", *and* exactly ℓ of $\alpha_2, \alpha_4, \dots, \alpha_{2n}$ are "$\mathtt{H}$", then

$$\mathsf{P}(\vec{\alpha}) = \left(\tfrac{3}{4}\right)^k \times \left(\tfrac{1}{4}\right)^{n-k} \times \left(\tfrac{1}{4}\right)^{\ell} \times \left(\tfrac{3}{4}\right)^{n-\ell} = \tfrac{3^{n+k-\ell}}{4^{2n}}.$$

For $1 \le i \le n$, let $\mathsf{FP}_i : \Omega \to \{0, 1\}$ be the ***random variable*** whose value is 1 if the first player hits the basket the i^{th} time they try to, and whose value is 0 otherwise: For $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_{2n}) \in \Omega$,

$$\mathsf{FP}_i(\vec{\alpha}) = \begin{cases} 1 & \text{if } \alpha_{2i-1} = \mathtt{H}, \\ 0 & \text{if } \alpha_{2i-1} = \mathtt{M}. \end{cases}$$

It follows by the above that $\mathsf{P}(\mathsf{FP}_i = 1) = \frac{3}{4}$, if $1 \le i \le n$.

(2 marks) (a) Give the definition for the ***expected value of a random variable, with respect to a probability distribution***.

Definition of "Expected Value":

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(1 mark) (b) Give the expected value of FP_i with respect to the above probability distribution P (for $1 \leq i \leq n$).

Expected Value of $\mathbf{FP}_i$***:***

For $1 \leq i \leq n$, let $\mathsf{SP}_i : \Omega \to \{0, 1\}$ be the random variable whose value is 1 if the *second* player hits the basket the i^{th} time they try to, and whose value is 0 otherwise: For $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_{2n}) \in \Omega$,

$$\mathsf{SP}_i = \begin{cases} 1 & \text{if } \alpha_{2i} = \mathtt{H}, \\ 0 & \text{if } \alpha_{2i} = \mathtt{M}. \end{cases}$$

(1 mark) (c) Give the expected value of SP_i with respect to the above probability distribution P (for $1 \leq i \leq n$).

Expected Value of $\mathbf{SP}_i$***:***

It can be shown that (for this probability distribution) the random variables

$$\mathsf{FP}_1, \mathsf{FP}_2, \dots, \mathsf{FP}_n, \mathsf{SP}_1, \mathsf{SP}_2, \dots, \mathsf{SP}_n$$

are *pairwise independent*. ***You may use this fact, without proving it, when answering the rest of this question.***

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Let $W : \Omega \to \mathbb{N}$ be the number of times the ball goes through the hoop — so that W is a random variable such that

$$W = \mathsf{FP}_1 + \mathsf{FP}_2 + \cdots + \mathsf{FP}_n + \mathsf{SP}_1 + \mathsf{SP}_2 + \cdots + \mathsf{SP}_n.$$

(2 marks) (d) Give the expected value of W and explain, briefly, why your answer is correct.

Expected Value of W:

Why Your Answer is Correct:

(4 marks) (e) Compute the ***variances*** of FP_i (for $1 \leq i \leq n$), SP_i (for $1 \leq i \leq n$), and W — showing enough work so that a marker can understand how your answers were obtained.

Variance of FP_i:

Variance of SP_i:

Variance of W:

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(3 marks)

(f) Give the best (that is, smallest) upper bound that you can for the probability that $W \geq \frac{3}{2}n$, when n is large — giving enough work so that a marker can understand how your answer was obtained.

Please make sure to name any results from lectures that you used when obtaining this bound.

Bound for Probability that $W \geq \frac{3}{2}n$:

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(8 marks)

2. Let $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$. Give a ***deterministic finite automaton*** whose language is the set L of strings $\omega \in \Sigma^\star$ such that ab appears at least *twice* as a substring of ω, that is, the language

$$L = \{\mu\mathtt{ab}\nu\mathtt{ab}\chi \mid \mu, \nu, \chi \in \Sigma^\star\}.$$

For each state q in your deterministic finite automaton, state the set of strings $S_q \subseteq \Sigma^\star$ whose processing ends in state q. In other words, if q_0 is the start state, and $\delta : Q \times \Sigma \to Q$ is the transition function, then

$$S_q = \{\omega \in \Sigma^\star \mid \delta^\star(q_0, \omega) = q\}.$$

You do not need to prove that your answer is correct. However, it must *be* readable, precise, and correct if full marks are to be received.

Deterministic Finite Automaton:

Corresponding Sets of Strings:

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(9 marks)

3. Let $\Sigma^\star$ be an alphabet and let $A, B \subseteq \Sigma^\star$ such that the language B is ***decidable***.

   Give a ***many-one reduction*** that could be used to prove the following.

   > ***Claim:*** If $A \cup B$ is ***undecidable*** then A is undecidable too.

   Note: Since B is decidable and $A \cup B$ is undecidable, these languages must be different. Thus there must exist some (fixed) string $x_{\text{Yes}} \in \Sigma^\star$ such that $x_{\text{Yes}} \in A$ and $x_{\text{Yes}} \notin B$. You may use this fact without proving it — and you may use this string, x_{Yes}, when defining your reduction.

   For full marks you should define the many-one reduction that you would use, clearly and correctly. You should also *list the properties* that you would need to prove for this reduction, if you gave a complete proof of the above claim. (You do not need to give a complete proof of the claim, or give proofs of the properties you list.)

   Something Else To Do If You are Stuck: In order to receive at most 5 marks, instead of at most 9 marks, you may prove that if $A \cup B$ is undecidable then A is undecidable, *without* using a many-one reduction — using some other kind of "closure property", instead. (*Yes:* You need to give a complete proof, if you use this option.)

   Which Are You Going To Do? That is: please say whether you are giving a many-one reduction, or not:

   The Rest of Your Answer:

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Part B: Attempt ***at most one*** of the following ***bonus questions***. If you start to answer them both and do not ***clearly*** show which should be graded then only the ***first*** of these will be read and marked.

(5 marks)

4. Consider the situation described in the first question. Consider the event "2W", that the second player gets more hoops than the first player does. That is, 2W is the set of outcomes
$$\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{2n}) \in \Omega$$
such that the size of the set
$$W_2 = \{i \in \mathbb{N} \mid 1 \leq i \leq n \text{ and } \alpha_{2i} = \mathtt{H}\}$$
is strictly greater than the size of the set
$$W_1 = \{i \in \mathbb{N} \mid 1 \leq i \leq n \text{ and } \alpha_{2i-1} = \mathtt{H}\}.$$
Give the best (that is, smallest) upper bound that you can for the probability of this event. For full marks your bound should approach 0 as n approaches $+\infty$.

Upper Bound for Probability that the Second Player Wins:

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(5 marks)

5. This question can only be answered if you gave a ***many-one reduction*** to answer Question #3.

   Prove the claim in this question *without* using a many-one reduction, as well. (That is, do what you were allowed to do "if you got stuck", as well.)

   Another Proof of the Claim in Question #3:

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(Extra Space if You Need More Room for an Answer)

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(Extra Space if You Need More Room for an Answer)