CPSC 351 — Tutorial Exercise #17
Additional Practice Problem

This problem will not be discussed during the tutorial, and a solution for this problem will not be made available. It can be used as a "practice" problem that can help you practice skills considered in the lecture presentation for Lectures #21, or in Tutorial Exercise #17.

1. Suppose, now, that Ω is a ***finite*** sample space, $\mathsf{P} : \Omega \to \mathbb{R}$ is a probability distribution, and $X : \Omega \to \mathbb{R}$. Let $\mu = \mathsf{E}[X]$ and let $\sigma^2 = \mathsf{var}(X)$.

   Let $Y : \Omega \to \mathbb{R}$ such that $Y = X - \mu$. That is, $Y(\alpha) = X(\alpha) - \mu$ for every outcome $\alpha \in \Omega$.

   (a) Prove that $\mathsf{E}[Y] = 0$.

   (b) Prove that $\mathsf{var}(Y) = \sigma^2$ (so that $\mathsf{var}(Y) = \mathsf{var}(X)$).

   (c) Prove that $\mathsf{E}[Y^2] = \sigma^2$ as well.

   Now let $a, b \in \mathbb{R}$ such that $a > 0$ and $b \geq 0$.

   (d) Prove that

   $$\begin{aligned} \mathsf{P}(Y \geq a) &= \mathsf{P}(Y + b \geq a + b) \\ &\leq \mathsf{P}((Y+b)^2 \geq (a+b)^2) \\ &\leq \frac{\mathsf{E}[(Y+b)^2]}{(a+b)^2} \end{aligned}$$

   Hint: Notice that $Y + b$ and $(Y + b)^2$ can also be considered to be random variables. Consider the use of other results from the preparatory reading for this lecture.

   (e) Consider what this means, when $b = \frac{\sigma^2}{a}$, in order to complete a proof of ***Cantelli's Inequality***.