# CPSC 351 — Tutorial Exercise #17
# Tail Bounds

The goal of this exercise is to give you more practice in computing tail bounds for random variables.

## Problems To Be Solved

1. Consider an experiment in which you roll a die (which could show any of the integers between $1$ and $6$), $n$ times, for some positive integer $n$. Since you are interested in the sequence of values rolled, the ***sample space*** is the set

$$\Omega = \{(\alpha_1, \alpha_2, \dots, \alpha_n) \mid \alpha_i \in \{1, 2, 3, 4, 5, 6\} \text{ for } 1 \leq i \leq n\}.$$

   Here, the sequence $(\alpha_1, \alpha_2, \dots, \alpha_n)$ represents the outcome where the die shows $\alpha_1$ the first time it is rolled, $\alpha_2$ the second time it is rolled, and so on.

   Note that $|\Omega| = 6^n$. Let us suppose that this is a fair die (each possible value is showed with the same probability), and no roll of the die depends on any of the others. We will therefore use a ***probability distribution*** $\mathsf{P} : \Omega \to \mathbb{R}$ such that $\mathsf{P}(\mu) = \frac{1}{|\Omega|} = 6^{-n}$, for each outcome $\mu \in \Omega$.

   Consider the number of times that "$6$" is rolled. This is the value of a random variable

$$X : \Omega \to \mathbb{N}$$

   such that, an outcome $\mu = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \Omega$,

$$X(\mu) = X((\alpha_1, \alpha_2, \dots, \alpha_n)) = |\{i \in \mathbb{N} \mid 1 \leq i \leq n \text{ and } \alpha_i = 6\}|.$$

(a) For $1 \le i \le n$, consider another random variable $X_i : \Omega \to \{0, 1\}$ such that, for $\mu = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \Omega$,

$$X_i(\mu) = X_i((\alpha_1, \alpha_2, \ldots, \alpha_n)) = \begin{cases} 1 & \text{if } \alpha_i = 6, \\ 0 & \text{otherwise.} \end{cases}$$

Describe, as precisely as you can, how $X_1, X_2, \ldots, X_n$ and $X$ are related.

(b) Compute E$[X_i]$ and var$(X_i)$, for each integer $i$ such that $1 \le i \le n$.

(c) Show that the random variables $X_1, X_2, \ldots, X_n$ are ***pairwise independent***.

(d) Using the above information, compute E$[X]$ and var$(X)$.

(e) Using the above information, and inequalities from lectures, find the best bounds that you can for the probability that $X < \frac{n}{12}$ as well as the probability that $X > \frac{n}{2}$ — ***based on what you know, so far***.

(f) Now show that the random variables $X_1, X_2, \ldots, X_n$ are ***mutually independent***.

(g) Using this additional information, find a better bound for the probability that $X < \frac{n}{12}$ when $n$ is large.