CPSC 351 — Tutorial Exercise #16
Random Variables and Expectation

The goal of this exercise is to give you more practice solving problems that concern conditional random variables and expectation.

Problems To Be Solved

1. Once again, consider an experiment in which you are tossing three coins: The sample space is

$$\Omega_3 = \{(\mathtt{H},\mathtt{H},\mathtt{H}),(\mathtt{H},\mathtt{H},\mathtt{T}),(\mathtt{H},\mathtt{T},\mathtt{H}),(\mathtt{H},\mathtt{T},\mathtt{T}),(\mathtt{T},\mathtt{H},\mathtt{H}),(\mathtt{T},\mathtt{H},\mathtt{T}),(\mathtt{T},\mathtt{T},\mathtt{H}),(\mathtt{T},\mathtt{T},\mathtt{T})\},$$

so that $|\Omega_3| = 2^3 = 8$.

Let X be the random variable "number of coins tossed until a head is flipped", with this set to be 4 if head is never flipped at all — so that, for $\vec{\alpha} \in \Omega_3$,

- $X(\vec{\alpha}) = 1$ if $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, where $\alpha_1 = \mathtt{H}$;
- $X(\vec{\alpha}) = 2$ if $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, where $\alpha_1 = \mathtt{T}$ and $\alpha_2 = \mathtt{H}$;
- $X(\vec{\alpha}) = 3$ if $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, where $\alpha_1 = \alpha_2 = \mathtt{T}$ and $\alpha_3 = \mathtt{H}$;
- $X(\vec{\alpha}) = 4$ if $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, where $\alpha_1 = \alpha_2 = \alpha_3 = \mathtt{T}$.

(a) Suppose that $\mathsf{P} : \Omega_3 \to \mathbb{R}$ is the ***uniform distribution***, so that $\mathsf{P}(\vec{\alpha}) = \frac{1}{8}$ for every element $\vec{\alpha}$ of Ω_3. Compute the expected value $\mathsf{E}[X]$ of X with respect to this probability distribution.

(b) Suppose that the coin is "biased" so that the probability of "heads" is $\frac{2}{3}$ each time the coin is tossed, instead of $\frac{1}{2}$ — and, in particular,

- $\mathsf{P}((\mathtt{H},\mathtt{H},\mathtt{H})) = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$,
- $\mathsf{P}((\mathtt{H},\mathtt{H},\mathtt{T})) = \mathsf{P}((\mathtt{H},\mathtt{T},\mathtt{H})) = \mathsf{P}((\mathtt{T},\mathtt{H},\mathtt{H}) = \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{3}\right) = \frac{4}{27}$,
- $\mathsf{P}((\mathtt{H},\mathtt{T},\mathtt{T})) = \mathsf{P}((\mathtt{T},\mathtt{H},\mathtt{T})) = \mathsf{P}((\mathtt{T},\mathtt{T},\mathtt{H})) = \left(\frac{2}{3}\right) \times \left(\frac{1}{3}\right)^2 = \frac{2}{27}$, and
- $\mathsf{P}((\mathtt{T},\mathtt{T},\mathtt{T})) = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$.

Compute the expected value $\mathsf{E}[X]$ of X with respect to *this* probability distribution.

(c) In order to generalize this, let p be a real number such that $0 \le p \le 1$, and suppose that the biased coin tosses "heads" with probability p — so that, in particular,

- $\mathsf{P}((\mathtt{H},\mathtt{H},\mathtt{H})) = p^3$,
- $\mathsf{P}((\mathtt{H},\mathtt{H},\mathtt{T})) = \mathsf{P}((\mathtt{H},\mathtt{T},\mathtt{H})) = \mathsf{P}((\mathtt{T},\mathtt{H},\mathtt{H}) = p^2 \times (1-p) = p^2 - p^3$,
- $\mathsf{P}((\mathtt{H},\mathtt{T},\mathtt{T})) = \mathsf{P}((\mathtt{T},\mathtt{H},\mathtt{T})) = \mathsf{P}((\mathtt{T},\mathtt{T},\mathtt{H})) = p \times (1-p)^2 = p - 2p^2 + p^3$, and
- $\mathsf{P}((\mathtt{T},\mathtt{T},\mathtt{T})) = (1-p)^3 = 1 - 3p + 3p^2 - p^3$.

Compute the expected value $\mathsf{E}[X]$ of X with respect to *this* probability distribution.

2. Recall that a random variable $X : \Omega \to \mathbb{R}$ is an ***indicator random variable*** if either $X(\mu) = 0$ or $X(\mu) = 1$ for every outcome $\mu \in \Omega$. If X is this kind of random variable, then we may associate the following ***event*** with X:

$$A_X = \{\mu \in \Omega \mid X(\mu) = 1\}$$

— noting that this is also the event that is sometimes denoted by "$X = 1$".

Prove that if X is an indicator random variable, A_X is as above, and the sample space, Ω, is finite, then

$$\mathsf{E}[X] = \mathsf{P}(A_X) = \mathsf{P}(X = 1).$$

3. Let Ω be a finite sample space, let $\mathsf{P} : \Omega \to \mathbb{R}$ be a probability distribution for Ω, let n be a positive integer, and let $X_1, X_2, \ldots, X_n$ be random variables for Ω, so that $X_i : \Omega \to \mathbb{R}$ for every integer i such that $1 \le i \le n$.

(a) Prove that $\mathsf{E}[X_1 + X_2] = \mathsf{E}[X_1] + \mathsf{E}[X_2]$.

(b) *Briefly* explain how you could use the above result to prove that

$$\mathsf{E}[X_1 + X_2 + \cdots + X_n] = \mathsf{E}[X_1] + \mathsf{E}[X_2] + \cdots + \mathsf{E}[X_n]$$

when $n \ge 3$, as well.