Lecture #19: Conditional Probability and Independence
Lecture Presentation

A Problem You Might Find in a Textbook

Suppose your sock drawer stores two pairs of ***red*** socks, three pairs of ***blue socks***, and four pairs of ***green*** socks. The socks of any given colour look the same, that each of the four red socks in the drawer "matches" each of the other red socks, each of the six blue socks "matches" each of the other blue socks, and each of eight green socks "matches" each of the other green socks. On the other hand, any two socks with different colours *do not* match.

In order to make defining a "sample space" easier, let's give these socks names: The four red socks are r_1, r_2, r_3 and r_4, the six blue socks are b_1, b_2, b_3, b_4, b_5 and b_6, and the eight green socks are g_1, g_2, g_3, g_4, g_5, g_6, g_7. g_8.

You like variety in life. When you get up and pull out two socks from your sock drawer, the first sock (which you will wear on your left foot) can be any one of the eighteen socks in the drawer, and each of the second sock (which you will wear on your right foot) can be any one of the seventeen socks that remain.

1. Describe the elements of the ***sample space*** Ω, corresponding to the "experiment" in which you choose socks to wear. (This should include, in total, exactly 18×17 ordered pairs of socks.)

   Sample Space:

Let us assume that each pair of socks is chosen with the same probability, that is, $\mathsf{P}(\alpha) = \frac{1}{18\times 17}$ for each element α (that is, choice of two socks) in your sample space.

Consider each of the following events:

- R: The first sock that you take from the drawer is ***red***.
- B: The first sock that you take from the drawer is ***blue***.
- G: The first sock that you take from the drawer is ***green***.
- M: The pair of socks, that you take from the drawer, ***match***.

2. Compute the conditional probability that the socks match, given that the first sock is ***red***.

Conditional Probability, Given That the First Sock is Red:

3. Compute the conditional probability that the socks match, given that the first sock is ***blue***.

   Conditional Probability, Given That the First Sock is Blue:

4. Compute the conditional probability that the socks match, given that the first sock is ***green***.

   Conditional Probability, Given That the First Sock is Green:

5. Use these conditional probabilities, and a result from Lecture #19 (which you should name), to compute the probability that the socks match.

   Probability That the Socks Match:

6. Prove that the following is true for *every* event A, for this sample space: if event A is ***attracted*** to R, then the same event A must be ***repelled by*** either event B, ***repelled by*** event G, or both.

   Proof of This Result:

A Problem Involving Hash Tables with Chaining

See the presentation for a Lecture #18 for a description of a ***hash table with chaining.*** Recall that this involves two parameters:

- The size, n, of the finite set S being represented. Let $k_1, k_2, \ldots, k_n$ be the elements of this set — and recall that these all belong to a large set called the "universe" $\mathcal{U}$. The set is not empty — so that n is a positive integer.
- The size, m of the hash table being used; m is also a positive integer.

Placement of the elements of S in the hash table depends on the choice of the ***hash function***

$$h : \mathcal{U} \rightarrow \{0, 1, \ldots, m-1\}$$

being used. Since the hash table (used to represent the set $S = \{k_1, k_2, \ldots, k_n\}$) only depends on the "hash values" $h(k_1), h(k_2), \ldots, h(k_n)$, each ***outcome*** can be represented as a sequence

$$(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

where $\alpha_i = h(k_i) \in \{0, 1, 2, \ldots, m-1\}$ for each integer i such that $1 \leq i \leq n$. That is, the ***sample space***, to be used here, is the set

$$\Omega_{n,m} = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \mid \alpha_i \in \mathbb{Z} \text{ and } 0 \leq i \leq m-1 \\ \text{for every integer } i \text{ such that } 1 \leq i \leq k\}.$$

As in the presentation for Lecture #17, we will be interested in each set

$$S_i = \{k_j \mid 1 \leq j \leq n \text{ and } h(k_j) = i\},$$

where i is an integer such that $0 \leq i \leq m-1$,

Let us use the ***uniform probability distribution***, that is, the function

$$\mathsf{P} : \Omega_{n,m} \rightarrow \mathbb{R}$$

such that

$$\mathsf{P}(\sigma) = \frac{1}{|\Omega_{n,m}|} = m^{-n}$$

for every outcome $\sigma \in \Omega_{n,m}$.

This question concerns the following events:

- A: $|S_1| = 2$.
- B: $h(k_1) = 1$.

1. What is the probability of event A?

   Probability of the Event A:

2. What is the conditional probability of event A, given event B?

   Conditional Probability of A, Given B:

3. For which choices of n and m is A ***attracted*** to B? For which choices of n and m is A ***repelled*** by B? For which choices of n and m is A ***indifferent*** to B?

   When is A Attracted to B? ...Repelled by B? ...Indifferent to B?

4. If possible, explain why the answer for the above question should not be surprising.

Why Does the Previous Answer Make Sense?

Proving a Result from the Required Reading

1. State and prove the ***Law of Total Probability***.

   The Law of Probability, and Its Proof: