CPSC 351 — Tutorial Exercise #8

Nonregular Languages

About This Exercise

This exercise is intended to help you to understand how to use the techniques that have been introduced, in this course, to prove that languages are not regular.

Getting Started

These initial questions concern ***mistakes*** that students often make when they try to use the techniques used in this exercise. There will probably not time to discuss these problems during the tutorial — but you should discuss these with the instructor, during the instructor's office hours, if you have questions about them.

1. Let $\Sigma = \{\mathrm{a}\}$ and let $L = \{\omega \in \Sigma^\star \mid \text{the length of } \omega \text{ is even}\}$. Please identify the ***error*** that has been made, in the following incorrect proof, as precisely as you can. (This error is similar to one that students have made, when writing proofs that languages are not regular, in the past.)

    Claim: The above language, L, is not regular.

    Proof. Suppose, to obtain a contradiction, that L is a regular language.
    Then it follows, by the Pumping Lemma for Regular Languages, that there is a number $p \geq 1$ such that if s is any string in L with length at least p, then s can be divided into three pieces, $s = xyz$ (for $x, y, z \in \Sigma^\star$) satisfying the following three conditions.

    1. $xy^iz \in L$ for every integer i such that $i \geq 0$.
    2. $|y| > 0$ (so that $y \neq \lambda$).
    3. $|xy| \leq p$.

Consider the string $s = a^{2p}$ where $p \geq 1$ is the number whose existence follows by the above claim.

- Since $2p$ is an even number $s \in L$.
- Since $p \geq 0$, $|s| = 2p \geq p$.

It now follows, by the above, that there exist strings $x, y, z \in \Sigma^\star$ such that $s = xyz$ and properties #1, #2 and #3, above, are satisfied.

With that noted, let $x = \lambda$, let $y = \mathsf{a}$, and let $z = \mathsf{a}^{2p-1}$. Then $|y| = 1 > 0$, so that property #2 is satisfied, and $|xy| = 1 \leq p$, so that property #3 is satisfied too. However, if $i = 0$ then i is an integer such that $i \geq 0$ and

$$xy^i z = xy^0 z = \lambda \cdot \lambda\lambda\mathsf{a}^{2p-1} = \mathsf{a}^{2p-1} \notin L,$$

since $|xy^i z| = 2p - 1$ is not an even number in this case. Thus property #1 is not satisfied and a ***contradiction*** has been obtained.

It follows that the original assumption must be false — and the above language L is not regular, as claimed. $\square$

2. Say whether each of the following statements is ***true*** or ***false***. Then write it using mathematical notation (so that the symbols "$\exists$" and "$\forall$" will be used); you may assume that it is "understood" that the types of values are integers here — you do not need to include this in the expressions that you write.

   (a) "For every integer x, there exists an integer y such that x is strictly less than y."

   (b) "There exists an integer y such that, for every integer x, x is strictly less than y".

   Note: The main difference between the sentences given in parts (a) and (b) is the order in which quantified variables are introduced. What (if anything) does suggest that something that you need to be careful about, when using results like "The Pumping Lemma for Regular Language", to prove things?

3. Let $\Sigma = \{\mathsf{a}, \mathsf{b}\}$ and let $L = \emptyset$. Please identify the ***error*** that has been made, in the following incorrect proof, as precisely as you can. (This error is similar to one that students have made, when writing proofs that languages are not regular, in the past.)

   Claim: The above language, L, is not regular.

   Proof. Recall that the languages

   $$L_1 = \{\mathsf{a}^n\mathsf{b}^n \mid n \in \mathbb{Z} \text{ and } n \geq 0\}$$

   and

   $$L_2 = \{\mathsf{a}^n\mathsf{b}^m \mid n \in \mathbb{Z}, m \in \mathbb{Z}, n, m \geq 0 \text{ and } n \neq m\}$$

   are languages such that $L_1 \subseteq \Sigma^\star$, $L_2 \subseteq \Sigma^\star$, L_1 is not regular, and L_2 is not regular. Since $L_1 \cap L_2 = \emptyset = L$ it follows that the language $L = \emptyset$ is not regular, as well. $\square$

Problems To Be Solved in the Tutorial

4. Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. Using the "Pumping Lemma for Regular Languages", prove that the language

$$L = \{\mathsf{a}^n\mathsf{b}^m \mid m < n\}$$

is not a regular language (where "$<$" represents the relation "strictly less than" so that, for example, it is not true that $3 < 3$).

The next problem is a bit more challenging — and requires you think, more carefully, about how to choose the number "i" that is used in a proof that uses the Pumping Lemma. It is OK if you are not able to solve it.

5. Let $\Sigma = \{\mathtt{a}\}$. Prove that the language

$$L = \{\mathsf{a}^n \mid n \text{ is a prime number}\} \subseteq \Sigma^\star$$

is not a regular language.

When solving the final problem — which involves closure properties — you may assume the result that you were asked to prove in the previous problem, even if you were not able to solve it.

6. Recall that a positive integer n is ***composite*** if $n \geq 2$ and there exist integers k and ℓ such that $2 \leq k, \ell \leq n - 1$ and $k \times \ell = n$. Every positive integer n, such that $n \geq 2$, is either prime or composite — but not both.

   Let $\Sigma = \{\mathtt{a}\}$ and let $L \subseteq \Sigma^\star$ be the language

$$L = \{\mathsf{a}^n \mid n \geq \mathbb{Z},\ n \geq 2, \text{ and } n \text{ is composite}\}.$$

   Use closure properties, and one or more languages that we have already shown not to be regular, to prove that L is not a regular language.