Lecture #7: Regular Operations and Closure Properties of Regular Language

Proofs of Closure Properties

Introduction

This document provides a proof of the following result — which was stated, but not proved, in the notes for Lecture #7.

Theorem 1. *Let Σ be an alphabet, and let $A, B \subseteq \Sigma^\star$.*

(a) If A and B are regular languages then $A \cup B$ is a regular language, as well.

(b) If A and B are regular languages, then $A \circ B$ is a regular language, as well.

(c) If A is a regular language then $A^\star$ is a regular language as well.

A Useful Minor Result

The following minor result will be repeatedly of use when developing a proof of the above claim.

Lemma 2. *Let Σ be an alphabet, and let $L \subseteq \Sigma^\star$. Then L is a regular language if and only if L is the language $L(M)$ of some nondeterministic finite automaton $M = (Q, \Sigma, \delta, q_0, F)$ which satisfies the following properties.*

(a) There are no transitions into q_0, at all. That is, $q_0 \notin \delta(q, \sigma)$ for any state $q \in Q$ or any symbol $\sigma \in \Sigma_\lambda$, so that the only string $\omega \in \Sigma^\star$ such that $q_0 \in \delta^\star(q_0, \omega)$ is the empty string, $\omega = \lambda$.

(b) M has exactly one accepting state, q_F, and there are no transitions out of this state. That is, $F = \{q_F\}$ and $\delta(q_F, \sigma) = \emptyset$ for every symbol $\sigma \in \Sigma_\lambda$.

Sketch of Proof. Suppose, first, that L is the language $L(M)$ of some nondeterministic finite automaton $M = (Q, \Sigma, \delta, q_0, F)$ which satisfies properties (i) and (ii), above. Then, since M

is a nondeterministic finite automaton, it follows by the results in established in Lecture #6 that L is the language of some deterministic finite automaton as well — that is, L is a regular language.

Suppose, next, that L is a regular language. Then — once again, by the results established in Lecture #6 — $L = L(\widehat{M})$ for some nondeterministic finite automaton

$$\widehat{M} = (\widehat{Q}, \Sigma, \widehat{\delta}, \widehat{q_0}, \widehat{F}).$$

Renaming the states in $\widehat{Q}$ if necessary, we may assume without loss of generality that $\widehat{Q}$ does not include states called either q_0 or q_F.

Consider an NFA $M = (Q, \Sigma, \delta, q_0, F)$ such that the following properties are satisfied.

- $Q = \widehat{Q} \cup \{q_0, q_F\}$ — that is, we have added states q_0 and q_F to the set of states of $\widehat{M}$.
- The only transition out of the new start state, q_0, is a λ-transition to the old start state $\widehat{q_0}$ of $\widehat{M}$. That is, $\delta(q_0, \lambda) = \{\widehat{q_0}\}$ and $\delta(q_0, \sigma) = \emptyset$ for every symbol $\sigma \in \Sigma$.
- Transitions for the states in $\widehat{Q}$ are unchanged — except that a λ-transition is added from each state in $\widehat{F}$ to the new state q_F. That is, $\delta(q, \sigma) = \widehat{\delta}(q, \sigma)$ for every state $q \in \widehat{Q}$ and symbol $\sigma \in \Sigma$, while if $q \in \widehat{Q}$ then

$$\delta(q, \lambda) = \begin{cases} \widehat{\delta}(q, \lambda) \cup \{q_F\} & \text{if } q \in \widehat{F}, \\ \widehat{\delta}(q, \lambda) & \text{if } q \notin \widehat{F}. \end{cases}$$

- q_F is the only accepting state of M — that is, $F = \{q_F\}$ — and there are no transitions out of q_F. That is, $\delta(q_F, \sigma) = \emptyset$ for all $\sigma \in \Sigma_\lambda$.

Using the above rules, the following properties about ***λ-closures of states*** are easily established.

- If $\lambda \notin L$ then the λ-closure of the new start state q_0 in M is the union of $\{q_0\}$ and the λ-closure of the old start state, $\widehat{q_0}$, in $\widehat{M}$.
- On the other hand, if $\lambda \in L$ then the λ-closure of the new start state q_0 in M is the union of $\{q_0, q_F\}$ and the λ-closure of the old start state, $\widehat{q_0}$, in $\widehat{M}$.
- For every state $q \in \widehat{Q}$, if the λ-closure of q in $\widehat{M}$ does not include any accepting states (that is, states in $\widehat{F}$), then the λ-closure of q in M is the same set as the λ-closure of q in $\widehat{M}$.
- For every state $q \in \widehat{Q}$, if the λ-closure of q in $\widehat{M}$ *does* include at least one accepting state, then the λ-closure of q in M is the union of the λ-closure of q in $\widehat{M}$ and the set $\{q_F\}$.

- The λ-closure of the new accepting state q_F in M is the set $\{q_F\}$

It follows by the above that

$$\delta^\star(q_0, \lambda) = \begin{cases} \{q_0, q_F\} \cup \widehat{\delta}^\star(\widehat{q}_0, \lambda) & \text{if } \lambda \in L, \\ \{q_0\} \cup \widehat{\delta}^\star(\widehat{q}_0, \lambda) & \text{if } \lambda \notin L, \end{cases}$$

so that $\lambda \in L(M)$ if and only if $\lambda \in L(\widehat{M})$. Furthermore, it can also be proved (by induction[1] on the length of the string ω) that if $\omega \in \Sigma$ is a *non-empty* string then

$$\delta^\star(q_0, \omega) = \begin{cases} \widehat{\delta}^\star(\widehat{q}_0, \omega) \cup \{q_F\} & \text{if } \omega \in L, \\ \widehat{\delta}^\star(\widehat{q}_0, \omega) & \text{if } \omega \notin L. \end{cases}$$

Thus $\omega \in L(M)$ if and only if $\omega \in L(\widehat{M})$ as well.

It follows that $L(M) = L(\widehat{M}) = L$ and, since M is a nondeterministic finite automaton that satisfies properties (a) and (b), above, this establishes the claim. □

Establishing Closure Under Union

Lemma 3. *Let Σ be an alphabet and let $L_1, L_2 \subseteq \Sigma^\star$. If L_1 and L_2 are both regular languages then $L_1 \cup L_2$ is a regular language as well.*

Sketch of Proof. Let Σ be an alphabet, let $L_1, L_2 \subseteq \Sigma^\star$, and suppose that the languages L_1 and L_2 are both regular. Then there exist nondeterministic finite automata

$$M_1 = \{Q_1, \Sigma, \delta_1, q_{1,0}, F_1\} \quad \text{and} \quad M_2 = \{Q_2, \Sigma, \delta_2, q_{2,0}, F_2\}$$

such that $L(M_1) = L_1$, $L(M_2) = L_2$, and these nondeterministic finite automata have all the properties described in Lemma 2 — so that, in particular, $F_1 = \{q_{1,F}\}$ for some state $q_{1,F} \in Q_1$ and $F_2 = \{q_{2,F}\}$ for some state $q_{2,F} \in Q_2$. Renaming states as needed we may assume that $Q_1 \cap Q_2 = \emptyset$ and that $q_0 \notin Q_1$ and $q_0 \notin Q_2$.

Now consider a nondeterministic finite automaton

$$M = (Q, \Sigma, \delta, q_0, F)$$

that has M_1 and M_2 as components and whose structure is as shown in Figure 1 on page 4. That is,

$$Q = \{q_0\} \cup Q_1 \cup Q_2,$$

[1] This proof, and other proofs by induction mentioned in this document, are left as exercises.

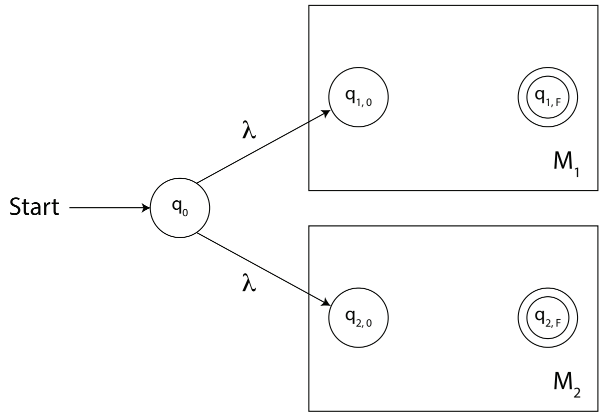

Figure 1: A Nondeterministic Finite Automaton with Language $L_1 \cup L_2$

the alphabet Σ is the same as for M_1 and M_2, the new state, q_0, is the start state,

$$F = F_1 \cup F_2 = \{q_{1,F}, q_{2,F}\},$$

and the transition function $\delta : Q \times \Sigma_\lambda \to \mathcal{P}(Q)$ is defined as follows.

- It is only possible to move from the new start state to one of the old start states, and no symbols are processed when doing this — so that

  $$\delta(q_0, \lambda) = \{q_{1,0}, q_{2,0}\}$$

  and

  $$\delta(q_0, \sigma) = \emptyset \quad \text{for every symbol } \sigma \in \Sigma.$$

- All transitions for states in Q_1 are the same in M as they were in M_1. That is,

  $$\delta(q, \sigma) = \delta_1(q, \sigma) \quad \text{for every state } q \in Q_1 \text{ and for all } \sigma \in \Sigma_\lambda.$$

- All transitions for states in Q_2 are the same in M as they were in M_2. That is,

  $$\delta(q, \sigma) = \delta_2(q, \sigma) \quad \text{for every state } q \in Q_2 \text{ and for all } \sigma \in \Sigma_\lambda.$$

This can be used to confirm that ***λ-closures*** in these automata are related as follows.

- The λ-closure of q in M is the union of $\{q_0\}$, the λ-closure of $q_{1,0}$ in M_1, and the λ-closure of $q_{2,0}$ in M_2.

- If $q \in Q_1$ (so that q is a state in the automaton M_1) then the λ-closure of q in M is the same set as the λ-closure of q in M_1.
- If $q \in Q_2$ (so that q is a state in the automaton M_2) then the λ-closure of q in M is the same set as the λ-closure of q in M_2.

It follows from the above that

$$\delta^\star(q_0, \lambda) = \{q_0\} \cup \delta_1^\star(q_{1,0}, \lambda) \cup \delta_2^\star(q_{2,0}, \lambda).$$

On the other hand, if ω is a non-empty string in $\Sigma^\star$ then it can be proved, by induction on the length of ω, that

$$\delta^\star(q_0, \omega) = \delta_1^\star(q_{1,0}, \omega) \cup \delta_2^\star(q_{2,0}, \omega).$$

Now, since $F = F_1 \cup F_2$ (and $q_0 \notin F$) it immediately follows that if $\omega \in \Sigma^\star$ then $\omega \in L(M)$ if and only if either $\omega \in L(M_1)$ or $\omega \in L(M_2)$ (or both). That is — since $L_1 = L(M_1)$ and $L_2 = L(M_2)$ —

$$L(M) = L_1 \cup L_2.$$

Since $L_1 \cup L_2$ is the language of a nondeterministic finite automaton it follows, by the results established in Lecture #6, that $L_1 \cup L_2$ is also the language of a *deterministic* finite automaton. That is, $L_1 \cup L_2$ is a regular language, as needed to establish the lemma. □

Establishing Closure Under Concatenation

Lemma 4. *Let Σ be an alphabet and let $L_1, L_2 \subseteq \Sigma^\star$. If L_1 and L_2 are both regular languages then $L_1 \circ L_2$ is a regular language as well.*

Sketch of Proof. Let Σ be an alphabet, let $L_1, L_2 \subseteq \Sigma^\star$, and suppose that the languages L_1 and L_2 are both regular. Then there exist nondeterministic finite automata

$$M_1 = \{Q_1, \Sigma, \delta_1, q_{1,0}, F_1\} \quad \text{and} \quad M_2 = \{Q_2, \Sigma, \delta_2, q_{2,0}, F_2\}$$

such that $L(M_1) = L_1$, $L(M_2) = L_2$, and these nondeterministic finite automata have all the properties described in Lemma 2 — so that, in particular, $F_1 = \{q_{1,F}\}$ for some state $q_{1,F} \in Q_1$ and $F_2 = \{q_{2,F}\}$ for some state $q_{2,F} \in Q_2$. Renaming states as needed we may assume that $Q_1 \cap Q_2 = \emptyset$ and that $q_0 \notin Q_1$ and $q_0 \notin Q_2$.

Now consider a nondeterministic finite automaton

$$M = (Q, \Sigma, \delta, q_0, F)$$

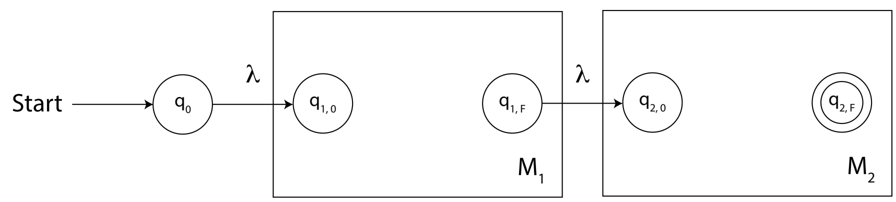

Figure 2: A Nondeterministic Finite Automaton with Language $L_1 \circ L_2$

that has M_1 and M_2 as components and whose structure is as shown in Figure 2, above. That is,

$$Q = \{q_0\} \cup Q_1 \cup Q_2,$$

the alphabet Σ is the same as for M_1 and M_2, the new state, q_0, is the start state,

$$F = F_2 = \{q_{2,F}\},$$

and the transition function $\delta : Q \times \Sigma_\lambda \to \mathcal{P}(Q)$ is defined as follows.

- It is only possible to move from the new start state to the start state for M_1, and no symbols are processed when doing this — so that

  $$\delta(q_0, \lambda) = \{q_{1,0}\}$$

  and

  $$\delta(q_0, \sigma) = \emptyset \quad \text{for every symbol } \sigma \in \Sigma.$$

- For every state $q \in Q_1$ such that $q \neq q_{1,F}$ (so that q is not M_1's accepting state)

  $$\delta(q, \sigma) = \delta_1(q, \sigma) \quad \text{for all } \sigma \in \Sigma_\lambda.$$

- It is possible to move from M_1's accepting state to M_2's start state, and no symbols are processed when doing so, so that

  $$\delta(q_{1,F}, \lambda) = \{q_{2,0}\}$$

  and

  $$\delta(q_{1,F}, \sigma) = \emptyset \quad \text{for every symbol } \sigma \in \Sigma.$$

- For every state $q \in Q_2$,

  $$\delta(q, \sigma) = \delta_2(q, \sigma) \quad \text{for all } \sigma \in \Sigma_\lambda.$$

This can be used to confirm that ***λ-closures*** in these automata are related as follows.

- If $\lambda \in L_1$ (so that $q_{1,F}$ is in the λ-closure of $q_{1,0}$ in M_1) then the λ-closure of q_0 in M is the union of $\{q_0\}$, the λ-closure of $q_{1,0}$ in M_1, and the λ-closure of $q_{2,0}$ in M_2.

  On the other hand, if $\lambda \notin L_1$, then the λ-closure of q_0 in M is the union of $\{q_0\}$ and the λ-closure of $q_{1,0}$ in M_1.

- For every state $q \in Q_1$, if $q_{1,F}$ is in the λ-closure of q in M_1, then the λ-closure of q in M is the union of the λ-closure of q in M_1 and the λ-closure of $q_{2,0}$ in M_2.

  On the other hand, if $q_{1,F}$ is *not* in the λ-closure of q in M_1, then the λ-closure of q in M is the same set as the λ-closure of q in M_1.

- For every state $q \in Q_2$, the λ-closure of q in M is the same set as the λ-closure of q in M_2.

It follows from the above that

$$\delta^\star(q_0, \lambda) = \begin{cases} \{q_0\} \cup \delta_1^\star(q_{1,0}, \lambda) \cup \delta_2^\star(q_{2.0}, \lambda) & \text{if } \lambda \in L_1, \\ \{q_0\} \cup \delta_1^\star(q_{1,0}, \lambda) & \text{if } \lambda \notin L_1. \end{cases}$$

The following properties can be established by induction on the length of the string, ω:

(a) For all states $r_1, r_2 \in Q_1$ and for every string $\omega \in \Sigma^\star$,

$$r_2 \in \delta^\star(r_1, \omega) \quad \text{if and only if} \quad r_2 \in \delta_1^\star(r_1, \omega).$$

(b) For every state $r_2 \in Q_1$ and for every string $\omega \in \Sigma^\star$,

$$r_2 \in \delta^\star(q_0, \omega) \quad \text{if and only if} \quad r_2 \in \delta_1^\star(q_{1,0}, \omega).$$

(c) For all states $r_1 \in Q_1$ and $r_2 \in Q_2$, $r_2 \in \delta^\star(r_1, \omega)$ if and only if there exist strings $\mu, \nu \in \Sigma^\star$ such that the following properties are satisfied.

   i. $\omega = \mu \cdot \nu$.
   ii. $q_{1,F} \in \delta_1^\star(r_1, \mu)$.
   iii. $r_2 \in \delta_2^\star(q_{2,0}, \nu)$.

(d) For every state $r_2 \in Q_2$, $r_2 \in \delta^\star(q_0, \omega)$ if and only if there exist strings $\mu, \nu \in \Sigma^\star$ such that the following properties are satisfied.

   i. $\omega = \mu \cdot \nu$.
   ii. $\mu \in L_1$ — so that $q_{1,F} \in \delta_1^\star(q_{1,0}, \mu)$.

iii. $r_2 \in \delta_2^\star(q_{2,0}, \nu)$.

(e) For all states $r_1 \in Q_2$ and $r_2 \in Q$,

$$r_2 \in \delta^\star(r_1, \omega) \quad \text{if and only if} \quad r_2 \in Q_2 \text{ and } r_2 \in \delta_2^\star(r_1, \omega).$$

Since $F = \{q_{2,0}\}$ it now follows by part (d), above, that — for every string $\omega \in \Sigma^\star$ — $\omega \in L(M)$ (that is, $q_{2,F} \in \delta^\star(q_0, \omega)$ if and only if there exist strings $\mu, \nu \in \Sigma^\star$ such that the following properties are satisfied

i. $\omega \in \mu \cdot \nu$.

ii. $\mu \in L_1$ — so that $q_{1,F} \in \delta_1^\star(q_{1,0}, \mu)$.

iii. $q_{2,F} \in \delta_2^\star(q_2, \nu)$ — so that $\nu \in L_2$.

That is, $L(M) = L_1 \circ L_2$.

Since $L_1 \circ L_2$ is the language of a nondeterministic finite automaton it follows, by the results established in Lecture #6, that $L_1 \circ L_2$ is also the language of a *deterministic* finite automaton. That is, $L_1 \circ L_2$ is a regular language, as needed to establish the lemma. ☐

Establishing Closure Under Kleene Star

Lemma 5. *Let Σ be an alphabet and let $L \subseteq \Sigma^\star$. If L is a regular language then $L^\star$ is a regular language as well.*

Sketch of Proof. Let Σ be an alphabet, let $L \subseteq \Sigma^\star$, and suppose that the language L is regular. Then there exists a nondeterministic finite automaton

$$M_1 = \{Q_1, \Sigma, \delta_1, q_{1,0}, F_1\}$$

such that $L(M_1) = L$, and this nondeterministic finite automaton has all the properties described in Lemma 2 — so that, in particular, $F_1 = \{q_{1,F}\}$ for some state $q_{1,F} \in Q_1$. Renaming states as needed we may assume that $q_0 \notin Q_1$.

Now consider a nondeterministic finite automaton

$$M = (Q, \Sigma, \delta, q_0, F)$$

that has M_1 as a component and whose structure is as shown in Figure 3 on page 9. That is,

$$Q = \{q_0\} \cup Q_1,$$

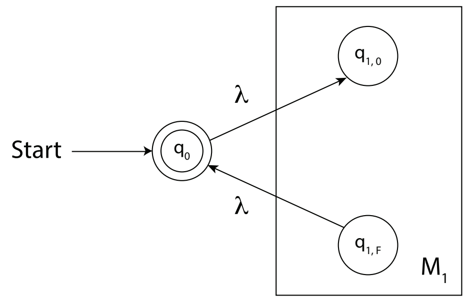

Figure 3: A Nondeterministic Finite Automaton with Language $L^\star$

the alphabet Σ is the same as for M_1, the new state, q_0, is the start state,

$$F = \{q_0\},$$

and the transition function $\delta : Q \times \Sigma_\lambda \to \mathcal{P}(Q)$ is defined as follows.

- It is only possible to move from q_0 to the start state, $q_{1,0}$, for M_1, and no symbols are processed when doing this — so that

  $$\delta(q_0, \lambda) = \{q_{1,0}\}$$

  and

  $$\delta(q_0, \sigma) = \emptyset \quad \text{for every symbol } \sigma \in \Sigma.$$

- For every state $q \in Q_1$ such that $q \neq q_{1,F}$,

  $$\delta(q, \sigma) = \delta_1(q, \sigma) \quad \text{for all } \sigma \in \Sigma_\lambda.$$

- It is only possible to move from $q_{1,F}$ to q_0, and no symbols are processed when doing that, so that

  $$\delta(q_{1,F}, \lambda) = \{q_0\}$$

  and

  $$\delta(q_{1,F}, \sigma) = \emptyset \quad \text{for every symbol } \sigma \in \Sigma.$$

This can be used to confirm that ***λ-closures*** in these automata are related as follows.

- The λ-closure of q_0 in M is the union of $\{q_0\}$ and the λ-closure of $q_{1,0}$ in M_1.

- For every state $q \in Q_1$, if $q_{1,F}$ belongs to the λ-closure of q in M_1, then the λ-closure of q in M is the union of the λ-closure of q in M_1, the set $\{q_0\}$, and the λ-closure of $q_{1,0}$ in M_1.

  On the other hand, if $q_{1,F}$ does not belong to the λ-closure of q in M_1, then the λ-closure of q in M is the same set as the λ-closure of q in M_1.

It follows from the above that

$$\delta^\star(q_0, \lambda) = \{q_0\} \cup \delta_1^\star(q_{0,1}, \lambda)$$

— so that $\lambda \in L(M)$, since $q_0 \in F$.

The following properties are satisfied for every non-empty string $\omega \in \Sigma^\star$ — and can be proved by mathematical induction on the length of ω:

(a) For every state $q \in Q_1$, $q \in \delta^\star(q_0, \omega)$ if and only if there exists an integer k such that $k \geq 0$, as well as strings $\mu_1, \mu_2, \dots, \mu_k, \nu \in \Sigma^\star$, such that the following properties are satisfied.

   i. μ_i is a non-empty string in $L = L(M_1)$ for every integer i such that $1 \leq i \leq k$.[2]

   ii. $q \in \delta_1^\star(q_{0,1}, \nu)$.

   iii. $\omega = \mu_1 \cdot \mu_2 \dots \mu_k \cdot \nu$.

(b) $q_0 \in \delta^\star(q_0, \omega)$ — so that $\omega \in L(M)$ — if and only if there exists a *positive* integer k, as well as strings $\mu_1, \mu_2, \dots, \mu_k \in \Sigma^\star$, such that the following properties are satisfied.

   i. μ_i is a non-empty string in $L = L(M_1)$ for every integer i such that $1 \leq i \leq k$.

   ii. $\omega = \mu_1 \cdot \mu_2 \dots \mu_k$.

It follows by the above that $L(M) = (L(M_1))^\star = L^\star$.

Since $L^\star$ is the language of a nondeterministic finite automaton it follows, by the results established in Lecture #6, that * is also the language of a *deterministic* finite automaton. That is, $L^\star$ is a regular language, as needed to establish the lemma. ☐

Completion of the Proof

Proof of Theorem 1. Part (a) and (b) of the claim are implied by Lemmas 3 and 4, respectively, with languages L_1 and L_2 (in the lemmas) replaced by A and B, respectively. Part (c) of the claim is implied by Lemma 5, with language L (in the lemma) replaced by A. ☐

[2]Note that this part of the claim is trivially satisfied when $k = 0$ because it is "vacuous" (that is, empty) — because there is no such integer i or string μ_i in this case, at all.