# Lecture #6: Equivalence of Deterministic Finite Automata and Nondeterministic Finite Automata

## A Bad Case for the Subset Construction

Near the end of the lecture notes, it was claimed that there exists an infinite sequence of languages

$$L_1, L_2, L_3, \dots \subseteq \Sigma^\star$$

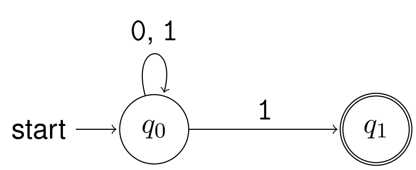
over the alphabet $\Sigma = \{0, 1\}$, such that — for every positive integer $k$ — $L_k$ is the language of a nondeterministic finite automaton with $k+1$ states. but such that every *deterministic* finite automaton with language $L_k$ must include at least $2^k$ states.

This document — which is for interest only (and is certainly not required reading) — presents a proof of this claim. It is based on material found in Section 2.3 of the text of Hopcroft, Motwani and Ullman [1].

As above, let $\Sigma = \{0, 1\}$, and let

$$L_1 = \{\omega \in \Sigma^\star \mid \omega \text{ ends with a } 1\}$$

Then the following nondeterministic finite automaton has language $L_1$:

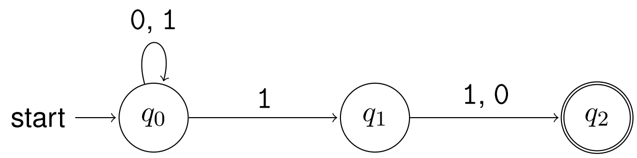



Languages $L_2, L_3, L_4, \dots \subseteq \Sigma^\star$ can be “inductively defined” by setting

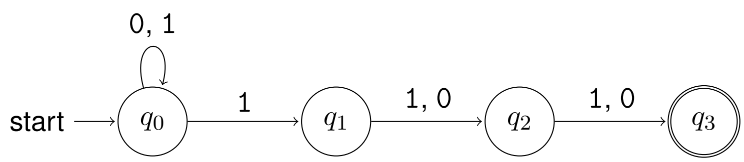
$$L_{k+1} = \{\omega \cdot \sigma \mid \omega \in L_k \text{ and } \sigma \in \Sigma\}.$$

Then $L_2$ includes all strings in $\Sigma^\star$ whose *second-to-last* symbol is 1... and so on.

Now, the following NFA has language $L_2$:




Similarly, the following NFA has language $L_3$:




For $k \geq 1$ consider an NFA $M_k = (Q_k, \Sigma, \delta_k, F_k)$ where

- $Q_k = \{q_0, q_1, q_2, \ldots, q_k\}$, so that $M_k$ has $k+1$ states.
- $\delta_k(q_0, 0) = \{q_0\}$, $\delta_k(q_0, 1) = \{q_0, q_1\}$, and $\delta_k(q_0, \lambda) = \emptyset$;
- For every integer $j$ such that $1 \leq j \leq k-1$, $\delta_k(q_j, 0) = \delta_k(q_j, 1) = \{q_{j+1}\}$ and $\delta_k(q_j, \lambda) = \emptyset$;
- $\delta_k(q_k, 0) = \delta_k(q_k, 1) = \delta_k(q_k, \lambda) = \emptyset$.
- $F_k = \{q_k\}$

Note that the nondeterministic finite automata, shown above, are the NFA's $M_2$ and $M_3$, respectively.

It is possible to prove the following (about $M_k$) by induction on $i$: For every integer $i$ such that $1 \leq i \leq k$, and for every string $\omega \in \Sigma^\star$,

$$q_i \in \delta^\star(q_0, \omega) \text{ if and only if } \omega \in L_i.$$

Thus $L(M_k) = L_k$ — so that $L_k$ has an NFA with only $k+1$ states.

**Claim 1.** *Let $\widehat{M} = (\widehat{Q}, \Sigma, \widehat{\delta}, \widehat{q}_0, \widehat{F})$ be a DFA such that $L(\widehat{M}) = L_k$. Then $|\widehat{Q}| \geq 2^k$, that is, $\widehat{M}$ has* at least $2^k$ *states.*

*Proof.* This will be proved ***by contradiction***. Let $k$ be a positive integer and suppose — to obtain a contradiction — that there exists a deterministic finite automaton

$$M = (Q, \Sigma, \delta, q_0, F)$$

with alphabet $\Sigma$, whose language is $M_k$, such that $|Q| < 2^k$ (that is, $M$ has strictly fewer than $2^k$ states).

$\Sigma^\star$ has *exactly* $2^k$ strings with length $k$ so it follows by the "Pigeonhole Principle" that there exist strings

$$\omega_1 = \sigma_1\sigma_2 \ldots \sigma_k \text{ and } \omega_2 = \tau_1\tau_2 \ldots \tau_k$$

in $\Sigma^\star$, both with length $k$, such that $\omega_1 \neq \omega_2$ but $\widehat{\delta}^\star(\widehat{q}_0, \omega_1) = \widehat{\delta}^\star(\widehat{q}_0, \omega_2)$.

Since $\omega_1 \neq \omega_2$ there is an integer $i$ such that $1 \leq i \leq k$ and $\sigma_i \neq \tau_i$. Without loss of generality we may assume that $\sigma_i = 1$ and $\tau_i = 0$ (we can just switch $\omega_1$ and $\omega_2$ otherwise). Then $\omega_1 \in L_{k-i+1}$ and $\omega_2 \notin L_{k-i+1}$

For $\ell \geq 0$ let $1^\ell$ denote a string of $\ell$ 1's — so that $1^0 = \lambda$, $1 = 1$, $1^2 = 11$, and so on.

Each of the following things can now be proved by induction on $\ell$: For every integer $\ell \geq 0$,

a) $\omega_1 \cdot 1^\ell \in L_{k+\ell-i+1}$ and $\omega_2 \cdot 1^\ell \notin L_{k+\ell-i+1}$ — so that (in particular, with $\ell = i-1$) $\omega_1 \cdot 1^{i-1} \in L_k$ and $\omega_2 \cdot 1^{i-1} \notin L_k$.

b) $\widehat{\delta}(\widehat{q}_0, \omega_1 \cdot 1^\ell) = \widehat{\delta}(\widehat{q}_0, \omega_2 \cdot 1^\ell)$ — so that (in particular, with $\ell = i-1$) $\widehat{\delta}(\widehat{q}_0, \omega_1 \cdot 1^{i-1})$ and $\widehat{\delta}(\widehat{q}_0, \omega_2 \cdot 1^{i-1})$ are both equal to the same state $\widehat{q} \in \widehat{Q}$.

Now, since $\omega_1 \cdot 1^{i-1} \in L_k$, $\widehat{\delta}(\widehat{q}_0, \omega_1 \cdot 1^{i-1}) = \widehat{q}$, and $L(\widehat{M}\,) = L_k$, it must be true that $\widehat{q} \in \widehat{F}$.

Since $\widehat{\delta}(\widehat{q}_0, \omega_2 \cdot 1^{i-1}) = \widehat{q}$ it now follows that $\omega_2 \cdot 1^{i-1} \in L(\widehat{M}\,) = L_k$ as well.

We have a ***contradiction*** — because we already know that $\omega_2 \cdot 1^{i-1} \notin L_k$.

So, an assumption that we made, along the way, must be incorrect. We only made one assumption, so *that* one must be false: "The DFA for $L_k$ being considered has fewer than $2^k$ states."

Since this was an arbitrarily chosen DFA whose language is $L_k$, it now follows that ***every*** DFA whose language is $L_k$ must have at least $2^k$ states, as claimed. $\square$

## References


[1] John E. Hopcroft, Rajeev Motwani, and Jeffrey D. Ullman. *Introduction to Automata Theory, Languages, and Computation*. Pearson Education, third edition, 2007.