Lecture #2: Introduction to Deterministic Finite Automata
Extended Transition Functions: Equivalence of Definitions

1 Introduction

During the first lecture on deterministic finite automata, *two* definitions were given for an extended transition function

$$\delta^\star : Q \times \Sigma^\star \to Q$$

for a deterministic finite automaton

$$M = (Q, \Sigma, \delta, q_0, F).$$

This document includes a ***proof*** that these definitions really are equivalent. CPSC 351 students are asked to read this because

- the equivalence of these definitions is quite important, and
- this document includes proofs of several properties, concerning strings, using ***mathematical induction***. Students in this course will be expected to read and understand proofs like this throughout the course — and may also be asked to *write* proofs like these on assignments.

2 Extended Transition Function — First Definition and Some Properties

As defined in the lecture notes, the ***extended transition function*** is a total function

$$\delta^\star : Q \times \Sigma^\star \to Q$$

such that if the automaton M is in state $q \in Q$, and the (sequence of symbols in) the string $\omega \in \Sigma^\star$ is received, then M is in state $\delta^\star(q, \omega)$ after that.

Thus

$$\delta^\star(q, \lambda) = q \tag{1}$$

for every state $q \in Q$.

Suppose, more generally, that

$$\omega = \sigma_1\sigma_2 \ldots \sigma_n \in \Sigma^\star$$

is a string with length $n \geq 0$ in $\Sigma^\star$ — so that, for $1 \leq i \leq n$, the i^{th} symbol in ω is σ_i.

Let $q \in Q$. Then a *sequence* of states $r_0, r_1, \ldots, r_n$, with length $n + 1$, can be defined as follows:

- $r_0 = q$, and
- $r_{i+1} = \delta(r_i, \sigma_{i+1})$ for every integer i such that $0 \leq i < n$.

In this case,

$$\delta^\star(q, \omega) = r_n, \tag{2}$$

the *last* state in the above sequence.

Note that the definitions given at lines (1) and (2) agree if $\omega = \lambda$ — for $n = 0$ in this case, so that $r_n = r_0 = q$.

The following *lemma* presents a property of sequences of states like the above that will be useful when properties of the extended transition function are considered.

Lemma 2.1. *Let $q \in Q$, let*

$$\mu = \sigma_1\sigma_2 \ldots \sigma_m \in \Sigma^\star \tag{3}$$

and

$$\nu = \tau_1\tau_2 \ldots \tau_n \in \Sigma^\star \tag{4}$$

be strings with lengths m and n respectively, and let

$$\omega = \mu \cdot \nu = \sigma_1\sigma_2 \ldots \sigma_m\tau_1\tau_2 \ldots \tau_n \in \Sigma^\star \tag{5}$$

be the concatenation of μ and ν, so that ω has length $n + m$.

Consider the sequences $r_0, r_1, \ldots, r_m$ and $s_0, s_1, \ldots, s_{m+n}$ of states, with lengths $m + 1$ and $m + n + 1$, respectively, that are defined using the following rules.

(a) $r_0 = q$.

(b) $r_{i+1} = \delta(r_i, \sigma_{i+1})$ *for every integer i such that $0 \leq i \leq m - 1$.*

(c) $s_0 = q$.

(d) $s_{i+1} = \delta(s_i, \sigma_{i+1})$ *for every integer i such that $0 \leq i \leq m - 1$.*

(e) $s_{m+j+1} = \delta(s_{m+j}, \tau_{j+1})$ *for every integer* j *such that* $0 \le j \le n-1$.

Then $r_i = s_i$ *for every integer* i *such that* $0 \le i \le m$.

Proof. It will be shown that $r_i = s_i$, for every integer i such that $0 \le i \le m$, by induction on i. The standard form of mathematical induction will be used.

Basis: if $i = 0$ then

$$\begin{aligned} r_i &= r_0 \\ &= q && \text{(by the equation in part (a) of the claim, above)} \\ &= s_0 && \text{(by the equation in part (c) of the claim, above)} \\ &= s_i \end{aligned}$$

as required to establish the claim in this case.

Inductive Step: Let k be an integer such that $k \ge 0$. It is necessary and sufficient to use the following

> *Inductive Hypothesis:* If $k \le m$ then $r_k = s_k$.

to prove the following

> *Inductive Claim:* If $k + 1 \le m$ then $r_{k+1} = s_{k+1}$.

There is nothing to prove if $k + 1 > m$. Suppose, therefore, that $k + 1 \le m$. Since $k \ge 0$ it follows that $0 \le k \le m - 1 \le m$, and it follows by the inductive hypothesis that $r_k = s_k$. Thus

$$\begin{aligned} r_{k+1} &= \delta(r_k, \sigma_{k+1}) && \text{(by part (b) of the above claim, since } 0 \le k \le m-1\text{)} \\ &= \delta(s_k, \sigma_{k+1}) && \text{(by the inductive hypothesis, as noted above)} \\ &= s_{k+1} && \text{(by part (d) of the above claim, since } 0 \le k \le m-1\text{)} \end{aligned}$$

as needed to complete the inductive step and to prove the claim. □

Note: The above claim is (arguably) so obvious that it probably does not need a proof. A proof has been given, anyway, to give an example of a proof of a property of a string using mathematical induction.

Lemma 2.2. *Let* $q \in Q$*, let* μ *be a string in* $\Sigma^\star$ *and let* τ *be a symbol in* Σ*. Then*

$$\delta^\star(q, \mu \cdot \tau) = \delta(\delta^\star(q, \mu), \tau).$$

Proof. This is a straightforward consequence of Lemma 2.1, for μ as in the claim (so that $m = |\mu|$), and when setting ν to be the string with length one whose first (and only) symbol is τ (so that $\tau_1 = \tau$ and $n = 1$). Then it follows by the definition of the extended transition function, given above, that $\delta^\star(q, \mu) = r_m$ — and, since

$$\mu \cdot \tau = \sigma_1 \sigma_2 \ldots \sigma_m \tau,$$

it also follows by the definition of the extended transition function that $\delta^\star(q, \mu \cdot \tau) = s_{m+n} = s_{m+1}$. Thus

$$\begin{aligned} \delta^\star(q, \mu \cdot \tau) &= s_{m+1} && \text{(as noted above)} \\ &= \delta(s_m, \tau) && \text{(by part (e) of the definition in Lemma 2.1, with } j = 0\text{)} \\ &= \delta(r_m, \tau) && \text{(since } s_m = r_m\text{, by Lemma 2.1)} \\ &= \delta(\delta^\star(q, \mu), \tau), && \text{(since } \delta^\star(q, \mu) = r_m\text{, as noted above)} \end{aligned}$$

as claimed. ☐

3 Extended Transition Function — Second Definition

Now consider the (possibly different) function

$$\delta^\times : Q \times \Sigma^\star \to Q$$

that corresponds to the *second* definition, That is, for every state $q \in Q$ and for every string $\omega \in \Sigma^\star$, let

$$\delta^\times(q, \omega) = \begin{cases} q & \text{if } \omega = \lambda, \\ \delta(\delta^\times(q, \mu), \tau) & \text{if } \omega = \mu \cdot \tau \text{ for } \mu \in \Sigma^\star \text{ and } \tau \in \Sigma. \end{cases} \tag{6}$$

4 Equivalence of Definitions

Theorem 4.1. *Let $\delta^\star : Q \times \Sigma^\star \to Q$ and $\delta^\times : Q \times \Sigma^\star \to Q$ be the functions defined in Sections 2 and 3, respectively. Then $\delta^\star = \delta^\times$, that is, $\delta^\star(q, \omega) = \delta^\times(q, \omega)$ for every state $q \in Q$ and for every string $\omega \in \Sigma^\star$.*

Proof. The result will be proved by induction on the length of the string ω mentioned in the claim. The standard form of mathematical induction will be used.

Basis: If $|\omega| = 0$ then $\omega = \lambda$, the empty string. It is therefore necessary and sufficient to prove that $\delta^\star(q, \lambda) = \delta^\times(q, \lambda)$ for every state $q \in Q$.

Suppose, therefore, that $q \in Q$. Then

$$\begin{aligned} \delta^\star(q, \lambda) &= q && \text{(by the equation at line (1))} \\ &= \delta^\times(q, \lambda), && \text{(by the equation at line (6))} \end{aligned}$$

as required.

Inductive Step: Let k be an integer such that $k \geq 0$. It is necessary and sufficient to use the following

> *Inductive Hypothesis:* $\delta^\star(q, \omega) = \delta^\times(q, \omega)$ for every state $q \in Q$ and for every string $\omega \in \Sigma^\star$ whose length is k.

to prove the following

> *Inductive Claim:* $\delta^\star(q, \omega) = \delta^\times(q, \omega)$ for every state $q \in Q$ and for every string $\omega \in \Sigma^\star$ whose length is $k + 1$.

With that noted, let q be a state in Q and let ω be a string in $\Sigma^\star$ with length $k + 1$. Then, since $k + 1 \geq 1$, there exists a string $\mu \in \Sigma^\star$ and a symbol $\tau \in \Sigma$ such that $\omega = \mu \cdot \tau$. Now $k+1 = |\omega| = |\mu| + 1$, so that the length of μ is k, and it follows by the inductive hypothesis that

$$\delta^\star(q, \mu) = \delta^\times(q, \mu).$$

Thus

$$\begin{aligned} \delta^\star(q, \omega) &= \delta^\star(q, \mu \cdot \tau) && \text{(since } \omega = \mu \cdot \tau\text{)} \\ &= \delta(\delta^\star(q, \mu), \tau) && \text{(by Lemma 2.2)} \\ &= \delta(\delta^\times(q, \mu), \tau) && \text{(by the inductive hypothesis, as noted above)} \\ &= \delta^\times(q, \mu \cdot \tau) && \text{(by the equation at line (6))} \\ &= \delta^\times(q, \omega) && \text{(since } \omega = \mu \cdot \tau\text{).} \end{aligned}$$

Since q and ω were arbitrarily chosen it follows that $\delta^\star(q, \omega) = \delta^\times(q, \omega)$ for *every* state $q \in Q$ and for *every* string $\omega \in \Sigma^\star$ whose length is $k + 1$ — as required to complete the inductive step, and the proof of the claim. $\square$

Thus the two definitions of an "extended transition function," given in the lecture notes, really are (provably) equivalent.