CPSC 351 — Tutorial Exercise #1
Review of Mathematical Foundations

About This Exercise

This exercise is based on material that you have ideally seen, and used, in one or more pre-requisites in this courses — and that will be needed later on. This material is summarized in the following documents.

- Math Review, Part One — Notation
- Math Review, Part Two — Definitions and Proofs
- Math Review, Part Three — Mathematical Induction

The Problem To Be Solved

Let $\mathbb{N}$ be the set of "natural numbers" — in this course, considered to be the set of non-negative integers, including 0. Consider the function $s : \mathbb{N} \to \mathbb{N}$ such that $s(0) = 0$, $s(1) = 1$, and, for $n \geq 2$,

$$s(n) = \sum_{i=0}^{n-1} s(i).$$

Prove that $s(n) = 2^{n-2}$ for every natural number n such that $n \geq 2$.

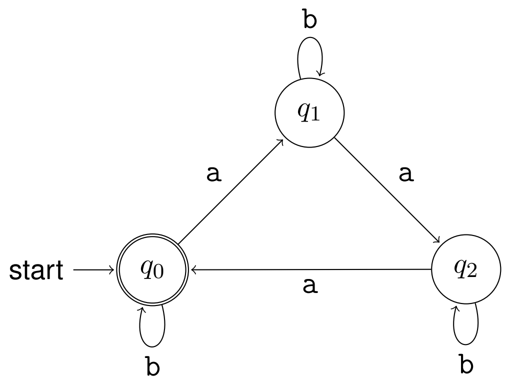

Figure 1: Deterministic Finite Automaton Considered in Lecture #2

A More Challenging Problem

The lecture presentation for Lecture #2 included a consideration of the ***deterministic finite automaton*** $M = (Q, \Sigma, \delta, q_0, F)$ such that $Q = \{q_0, q_1, q_2\}$, $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, q_0 is the start state, $F = \{q_0\}$, and the transition function $\delta : Q \times \Sigma \to Q$ is as shown in Figure 1, above.

During the lecture presentation, the following claim was introduced.

Theorem 1. *Let $n \in \mathbb{N}$. Then, for every string $\omega \in \Sigma^\star$ such that $|\omega| = n$, the following properties are satisfied.*

(a) $\delta^\star(q_0, \omega) = q_0$ if and only if the number of copies of "$\mathtt{a}$" in ω is congruent to 0 mod 3.

(b) $\delta^\star(q_0, \omega) = q_1$ if and only if the number of copies of "$\mathtt{a}$" in ω is congruent to 1 mod 3.

(c) $\delta^\star(q_0, \omega) = q_2$ if and only if the number of copies of "$\mathtt{a}$" in ω is congruent to 2 mod 3.

Suppose that you decided to prove this result by induction on n, using the ***standard form of mathematical induction***.

(a) State, as precisely as you can, the ***result*** that you would need to prove, in order to complete the ***basis*** in this proof.

If you are following the outline that has been used in this course, for this kind of proof, then the ***inductive step*** would probably begin with the following declaration of an integer variable:

> "Let k be an integer such that $k \geq 0$."

The ***Inductive Hypothesis*** that you could use in the Inductive Step, and the ***Inductive Claim*** that you would be proving in the Inductive Step, would both involve this variable, k.

(b) State the ***Inductive Hypothesis*** that would be used in the Inductive Step, as precisely as you can.

(c) State the ***Inductive Claim*** that would be proved in the Inductive Step, as precisely as you can.

Note: You do not really need to understand ***anything*** about deterministic finite automata in order to answer this question — because the questions really only concern how you use the standard form of mathematical induction, as a proof technique, to organize a proof.