CPSC 351 — Mathematics Review
Part Three: Mathematical Induction

Ideally, everything in the document should be a ***review*** of material that you learned about in a prerequisite course. It will be assumed that you understand and can use all of it in CPSC 351.

This document is based, heavily, on the presentation of mathematical induction found in Susanna S. Epp's text, *Discrete Mathematics with Applications* [1], which was the textbook used when some students completed the prerequisite course, MATH 271. It should also resemble the introduction to mathematical induction found in Kenneth H. Rosen's text, *Discrete Mathematics and Its Applications* [2], which was probably used as the textbook in CPSC 251, if you completed this prerequisite course instead.

The Standard Form of Mathematical Induction

Inductive Principle

The following "Inductive Principle" is a ***axiom*** this is useful for proving properties of the set of natural numbers — or the set of integers that are greater than or equal to a given initial value:

> Let $P(n)$ be a property that is defined for all integers n such that $n \geq \alpha$, for some integer α. Suppose the following two statements are true:
>
> 1. $P(\alpha)$ is true.
> 2. For all integers $k \geq \alpha$, if $P(k)$ is true then $P(k+1)$ is true.
>
> Then $P(n)$ is true for every integer $n \geq \alpha$.

A Corresponding Proof Technique

The following process — whose correctness is a consequence of the above axiom — can be used to prove that $P(n)$ is true for every integer n such that $n \geq \alpha$:

1. ***Basis:*** Show that $P(\alpha)$ is true.
2. To begin the ***Inductive Step***, introduce an integer k such that $k \geq \alpha$.[1]
3. To continue the ***Inductive Step***, assuming only the ***Inductive Step***, prove the ***Inductive Claim*** — where the "Inductive Step" and the "Inductive Claim" are as follows.

   Inductive Hypothesis: $P(k)$ is true,

   Inductive Claim: $P(k+1)$ is true.
4. Conclude that $P(n)$ is true for every integer $n \geq \alpha$.

Example

Problem To Be Solved. Suppose that we want to prove that

$$\sum_{i=0}^{n}(2i+1) = (n+1)^2$$

for every integer n such that $n \geq 0$.

Figuring out How to Use the Proof Technique.

- This is the kind of result that can be provided, using the above method, where $\alpha = 0$ and $P(n)$ is the property that $\sum_{i=0}^{n}(2i+1) = (n+1)^2$.
- Since $\alpha = 0$, we must show that $P(0)$ is true in order to complete the ***basis***. That is, we must show that

  $$\sum_{i=0}^{0}(2i+1) = (0+1)^2.$$

  The left hand side is the sum of a single term $2 \cdot 0 + 1 = 1$. The right hand side is $(0+1)^2 = 1^2 = 1$ as well, so we can either simplify both sides, to show that they are both equal to one — or (rewriting this argument) perform a sequence of algebraic operations in order to get from the expression on the left hand side to the expression on the right hand side (or vice-versa). *Any of these* approaches could be used to carry out this part of the proof.
- Since $\alpha = 0$, we should begin the inductive step by introducing an integer k such that $k \geq 0$.

[1] Other names for this integer, besides "k", can also be used.

- We must continue (and complete) the inductive step by using the following "Inductive Hypothesis" to prove the following "Inductive Claim".

  Inductive Hypothesis: $\displaystyle\sum_{i=0}^{k}(2i+1) = (k+1)^2.$

  Inductive Claim: $\displaystyle\sum_{i=0}^{k+1}(2i+1) = ((k+1)+1)^2.$

  Notice that the expression on the left hand side of the "Inductive Claim" is the sum shown on the left hand side of the "Inductive Hypothesis", plus a final term. Thus, if the sum on the left hand side of the inductive claim is split into pieces then the inductive hypothesis can be used to simplify one of these pieces. It turns out that simple algebraic manipulation is all that is needed to complete the inductive step, as shown below.

- The process should be completed by concluding the desired result (letting a reader know that we are finished and explaining why, if needed).

Solution. We wish to prove the following claim.

Theorem 1.

$$\sum_{i=0}^{n}(2i+1) = (n+1)^2$$

for every integer n such that $n \geq 0$.

Proof. The result will be proved by induction on n. The standard form of mathematical induction will be used.

Basis: Suppose that $n = 0$. Then

$$\begin{aligned}\sum_{i=0}^{n}(2i+1) &= \sum_{i=0}^{0}(2i+1)\\ &= (2 \cdot 0 + 1)\\ &= 0 + 1\\ &= 1,\end{aligned}$$

and

$$\begin{aligned}(n+1)^2 &= (0+1)^2\\ &= 1^2\\ &= 1.\end{aligned}$$

Since both sides are equal to 1,

$$\sum_{i=0}^{n}(2i+1) = (n+1)^2$$

in this case.

Inductive Step: Let k be an integer such that $k \geq 0$. It is necessary to use the following "Inductive Hypothesis" to prove the following "Inductive Claim".

Inductive Hypothesis: $\displaystyle\sum_{i=0}^{k}(2i+1) = (k+1)^2$.

Inductive Claim: $\displaystyle\sum_{i=0}^{k+1}(2i+1) = ((k+1)+1)^2$.

Since $k \geq 0$, $k+1 \geq 1$. Thus

$$\begin{aligned}
\sum_{i=0}^{k+1}(2i+1) &= \left(\sum_{i=0}^{k}(2i+1)\right) + 2\cdot(k+1)+1 && \text{(splitting off the final term)}\\
&= (k+1)^2 + 2\cdot(k+1)+1 && \text{(by the Inductive Hypothesis)}\\
&= (k^2+2k+1)+(2k+2)+1\\
&= k^2+4k+4 && \text{(rearranging terms)}\\
&= (k+2)^2\\
&= ((k+1)+1)^2;
\end{aligned}$$

that is, the "Inductive Claim" holds, as needed to complete the Inductive Step.

It now follows, by induction on n, that

$$\sum_{i=0}^{n}(2i+1) = (n+1)^2$$

for every integer n such that $n \geq 0$, as claimed. ☐

The Strong Form of Mathematical Induction

Inductive Principle

The following "Inductive Principle" is also an ***axiom*** that is useful for proving properties of the set of natural numbers — or the set of integers that are greater than or equal to a given initial value.

> Once again, let $P(n)$ be a property that is defined for all integers n. Let α and β be fixed integers such that $\alpha \leq \beta$. Suppose that the following two statements are true.
>
> 1. $P(\alpha), P(\alpha+1), P(\alpha+2), \dots, P(\beta)$ are all true.
> 2. For every integer $k \geq \beta$, if $P(i)$ is true for every integer i such that $\alpha \leq i \leq k$, then $P(k+1)$ is true as well.
>
> Then $P(n)$ is true for every integer $n \geq \alpha$.

A Corresponding Proof Technique

The following process — whose correctness is a consequence of the above axiom — can be used to prove that $P(n)$ is true for every integer n such that $n \geq \alpha$:

1. ***Choice of Breakpoint:*** Choose an integer β such that $\beta \geq \alpha$.[2]
2. ***Basis:*** Prove that $P(\alpha), P(\alpha+1), P(\alpha+2), \dots, P(\beta)$ are all true.
3. To begin the ***Inductive Step***, introduce an integer k such that $k \geq \beta$.[3]
4. To continue the ***Inductive Step***, assuming only the ***Inductive Step***, prove the ***Inductive Claim*** — where the "Inductive Step" and the "Inductive Claim" are as follows.

   Inductive Hypothesis: $P(i)$ is true for every integer i such that $\alpha \leq i \leq k$.

   Inductive Claim: $P(k+1)$ is true.

5. Conclude that $P(n)$ is true for every integer $n \geq \alpha$.

Example

Problem To Be Solved. Suppose that $g_0, g_1, g_2, \dots$ are integers such that (for every non-negative integer i)

$$g_i = \begin{cases} 12 & \text{if } i = 0, \\ 29 & \text{if } i = 1, \\ 5 \cdot g_{i-1} - 6 \cdot g_{i-2} & \text{if } i \geq 2. \end{cases}$$

[2]Breakpoints will not be called this very much later on. Instead, something like the phrase "The cases that $\alpha \leq i \leq \beta$ will be considered in the basis." will appear near the beginning of the proof.

[3]Once again, other names for this integer, besides "k", can also be used.

Consider the problem of proving that $g_n = 5 \cdot 3^n + 7 \cdot 2^n$ for every integer $n \geq 0$.

Figuring out How to Use the Proof Technique.

- if the sequence of integers $g_0, g_1, g_2, \ldots$ is as defined above, then this is the kind of result that can be proved, using the above method, where $\alpha = 0$ and $P(n)$ is the property that $g_n = 5 \cdot 3^n + 7 \cdot 2^n$.

- Notice that g_n is defined differently, when $n = 0$ or $n = 1$, than it is defined when $n \geq 2$. Furthermore, if $n \geq 2$ then the definition of g_n refers to both g_{n-1} and g_{n-2}. Let us choose the "breakpoint"' β to be 1, so that the special cases $n = 0$ and $n = 1$ are both considered in the basis, while the more general case, that $n \geq 2$, will be (only) considered in the inductive step.

  It would *not* be a mistake β differently here, but the inductive step would end up being more complicated if β was chosen to be 0, and the proof would probably be a bit more repetitive than necessary, if you chose β to be greater than or equal to 2.

- Since $\alpha = 0$ and $\beta = 1$, we must show that $P(0)$ and $P(1)$ are both true in order to complete the ***basis***. That is, we must show that

  $$g_0 = 5 \cdot 3^0 + 7 \cdot 2^0$$

  and that

  $$g_1 = 5 \cdot 3^1 + 7 \cdot 2^1.$$

  It will turn out that each of these can be established by using the definition of g_n, when $n = 0$, and $n = 1$, and carrying out a little bit of algebraic manipulation, to shown that both equations hold, because the left hand side and the right hand side can be shown to be equal to the same value.

- Since $\beta = 1$, we should begin the inductive step by introducing an integer k such that $k \geq 1$.

- Since $\alpha = 0$, we must continue (and complete) the inductive step by using the following "Inductive Hypothesis" to prove the following "Inductive Claim".

  Inductive Hypothesis: $g_i = 5 \cdot 3^i + 7 \cdot 2^i$ for every integer i such that $0 \leq i \leq k$.

  Inductive Claim: $g_{k+1} = 5 \cdot 3^{k+1} + 7^{k+1}$.

  Since $k \geq 1$, $k + 1 \geq 2$, so it follows by the definition of the sequence $g_0, g_1, g_2, \ldots$ that $g_{k+1} = 5 \cdot g_k - 6 \cdot g_{k-1}$. Since $k \geq 1$, $0 \leq k - 1, k \leq k$, and the inductive hypothesis can be used to conclude that $g_k = 5 \cdot 3^k + 7 \cdot 2^k$ and that $g_{k-1} = 5 \cdot 3^{k-1} + 7 \cdot 2^{k-1}$. All that is needed is a little bit of algebraic manipulation of expressions to establish the inductive claim, after that.

- Once again, the process can be completed by concluding the desired result (letting a reader know that we are finished and explaining why, if needed).

Solution: Suppose, once again, that $g_0, g_1, g_2, \dots$ is a sequence of integers such that (for every non-negative integer i)

$$g_i = \begin{cases} 12 & \text{if } i = 0, \\ 29 & \text{if } i = 1, \\ 5 \cdot g_{i-1} - 6 \cdot g_{i-2} & \text{if } i \geq 2. \end{cases}$$

We wish to prove the following claim.

Theorem 2. *$g_n = 5 \cdot 3^n + 7 \cdot 2^n$ for every integer n such that $n \geq 0$.*

Proof. This will be proved by induction on n. The strong form of mathematical induction will be used, and the cases that $n = 0$ and $n = 1$ will be considered in the basis.

Basis: Suppose, first, that $n = 0$. Then

$$\begin{aligned} g_n &= g_0 \\ &= 12 \end{aligned}$$

by the definition given for the sequence $g_0, g_1, g_2, \dots$, while

$$\begin{aligned} 5 \cdot 3^n + 7 \cdot 2^n &= 5 \cdot 3^0 + 7 \cdot 2^0 \\ &= 5 \cdot 1 + 7 \cdot 1 \\ &= 5 + 7 \\ &= 12 \end{aligned}$$

as well. Thus $g_n = 5 \cdot 3^n + 7 \cdot 2^n$ when $n = 0$.

Suppose, next, that $n = 1$. Then

$$\begin{aligned} g_n &= g_1 \\ &= 29 \end{aligned}$$

by the definition given for the sequence $g_0, g_1, g_2, \dots$, while

$$\begin{aligned} 5 \cdot 3^n + 7 \cdot 2^n &= 5 \cdot 3^1 + 7 \cdot 2^1 \\ &= 5 \cdot 3 + 7 \cdot 2 \\ &= 15 + 14 \\ &= 29 \end{aligned}$$

as well. Thus $g_n = 5 \cdot 3^n + 7 \cdot 2^n$ when $n = 1$ (and $g_n = 5 \cdot 3^n + 7 \cdot 2^n$ for *every* integer n such that $0 \le n \le 1$).

Inductive Step: Let k be an integer such that $k \ge 1$; it is necessary to use the following "Inductive Hypothesis" to prove the following "Inductive Claim".

Inductive Hypothesis: $g_i = 5 \cdot 3^i + 7 \cdot 2^i$ for every integer i such that $0 \le i \le k$.

Inductive Claim: $g_{k+1} = 5 \cdot 3^{k+1} + 7^{k+1}$.

Since $k \ge 1$, $k + 1 \ge 2$. Therefore

$$
\begin{aligned}
g_{k+1} &= 5 \cdot g_{(k+1)-1} - 6 \cdot g_{(k+1)-2} && \text{(by the definition of the sequence } g_0, g_1, g_2, \dots\text{)} \\
&= 5 \cdot g_k - 6 \cdot g_{k-1} \\
&= 5 \cdot (5 \cdot 3^k + 7 \cdot 2^k) - 6 \cdot (5 \cdot 3^{k-1} + 7 \cdot 2^{k-1}) \\
& && \text{(by the inductive hypothesis, since } 0 \le k-1, k \le k\text{)} \\
&= 5 \cdot (5 \cdot 3 - 6) \cdot 3^{k-1} + 7 \cdot (5 \cdot 2 - 6) \cdot 2^{k-1} && \text{(rearranging terms)} \\
&= 5 \cdot (15 - 6) \cdot 3^{k-1} + 7 \cdot (10 - 6) \cdot 2^{k-1} \\
&= 5 \cdot 9 \cdot 3^{k-1} + 7 \cdot 4 \cdot 2^{k-1} \\
&= 5 \cdot 3^{k+1} + 7 \cdot 2^{k+1} && \text{(since } 9 = 3^2 \text{ and } 4 = 2^2\text{).}
\end{aligned}
$$

That is, the "Inductive Claim" holds, as needed to complete the Inductive Step.

It now follows, by induction on n, that $g_n = 5 \cdot 3^n - 6 \cdot 3^n$ for every integer n such that $n \ge 0$, as claimed. ☐

A Note Concerning the Examples

You will not be expected to include material to show how you figured out "how to use the proof technique", like the above, when solving problems for assignments and tests, This material was just provided above, to try to make it clearer that each proof technique *was* being applied to produce the solution that was presented.

You also do not need to include quite as many details as might be suggested, above — this is (to some extent) a matter of judgment and personal taste. I find that when a class is sufficiently large, students will ask how I got from one step to another, if I *do not* include all the details in a written proof — so I generally include them.

With that noted, you should remember that ***markers cannot read your mind*** and ***markers are not expected to answer questions for you***. Students sometimes make the mistake of leaving so much out — including definitions of technical terms they have made up or variables they are using, identification of the process they are using, and even a statement of what they are trying to prove — that their answers are impossible to understand. Please make sure that you always include enough material so that your answer can be considered to be acceptably complete: You should never ask a marker to complete a *major* step for you, and you should never expect a marker to assume that you understand material or know how to do something, when other a significant number of the other students in the class might not understand or know how to do this.

References

[1] Susanna S. Epp. *Discrete Mathematics with Applications*. Brooks Cole, fifth edition, 2019.

[2] Kenneth H. Rosen. *Discrete Mathematics and Its Applications*. McGraw-Hill Education, eighth edition, 2018.