CPSC 351 — Mathematics Review
Part One: Notation

Ideally, everything in the document should be a ***review*** of material that you learned about in a prerequisite course. It will be assumed that you understand and can use all of it in CPSC 351.

This document is based, heavily, on the review included in Sipser's text, *Introduction to the Theory of Computation* [3]. It should also be consistent with material in the textbook in the course in discrete mathematics that you completed as a prerequisite for this one — probably (but not necessarily) either the text of Epp [1] or Rosen [2].

Sets

Set Notation

A ***set*** is a collection of objects. Sets may contain any type of object, including numbers, symbols, or other sets.

Finite Sets. If a set has only a finite number of members, then it can be given by listing its members, separated by commas and enclosed by braces — so that, for example,

$$\{1, 3, 6, 10\}$$

is one way to describe the set of, size four, whose members are 1, 3, 6 and 10. The ordering in which elements are listed does not matter, and it also does not matter if elements are listed more than once — so that

$$\{1, 6, 10, 3\}$$

and

$$\{6, 6, 10, 3, 1, 3, 1\}$$

are also ways to describe the same set.

Thus an ***empty set*** — one with no members — can be represented by a pair of braces with nothing between them,

$$\{\}$$

However, the symbol $\emptyset$ is often used to represent an empty set too.

A "set of size one" and the object contained in it are not the same thing! For example, 1 is different from $\{1\}$ — the set of size one that has the number 1 as its only member.

Infinite Sets. Several infinite sets of numbers have their own symbols. Students should be familiar with the set, $\mathbb{R}$, of ***real numbers***, and the set, $\mathbb{Z}$, of ***integers*** (including positive whole number, negative whole numbers, and zero).

The symbol $\mathbb{N}$ is used to denote the set of ***natural numbers***, which includes all of the positive integers and none of the negative integers. Some references define $\mathbb{N}$ so that 0 is a natural number, while others do not; in this course, 0 will generally considered to be a natural number (but this will be clarified for students when this is needed to understand course material). The symbol $\mathbb{P}$ might be used to represent the set of *positive* integers, such that

$$\mathbb{N} = \mathbb{P} \cup \{0\}.$$

It is possible to describe a subset of a larger set U (where U is sometimes called a ***universe***) by including a *condition* (that is, *property*) that the members of the set must satisfy. For example,

$$\{n \in \mathbb{Z} \mid n \text{ is even}\}$$

describes the set

$$\{\cdots - 6, -4, -2, 0, 2, 4, 6 \ldots\}$$

of integers that are even (that is, divisible by two). In general, the condition on the left ("$n \in \mathbb{Z}$", for this example) identifies the universe from which the members of set are chosen (in this example, $\mathbb{Z}$); "|" is used as separator; the statement to the right of the separator ("n is even", for this example) is the condition that the elements of the set must satisfy. In this course, this notation will almost always be used to describe *infinite* sets.

Operations on Sets

The symbol "$\in$" is used to denote ***set membership***. That is, if a is an object that might belong to some set S, then the statement

$$a \in S$$

is used to state that a is a member of the set S. Thus, if $S = \{0, 3, 5\}$ then each of the statements "$0 \in S$", "$3 \in S$", and "$5 \in S$" are true. On the other hand, the statement

$$a \notin S$$

is used to state that a is *not* a member of S — so that, for this example, the statement "$3 \notin S$" is false and the statement "$6 \notin S$" is true.

On the other hand, the symbol "$\subseteq$" is used to denote the fact that one set is a ***subset*** of another. That is, if S and T are sets then the statement

$$S \subseteq T$$

is used to state that every member of the set S is also a member of the set T — so that, for example,

$$\{0, 3\} \subseteq \{0, 1, 3, 6\}$$

and

$$\{0, 3\} \subseteq \mathbb{Z}$$

are true statements. Here are some related symbols that are used to state relationships between sets, and their meanings:

- "$\subset$" — For sets S and T, $S \subset T$ if it is true that $a \in S$ for every object a such that $a \in T$ (so that $S \subseteq T$), but there also exists at least one object b such that $b \in T$ and $b \notin S$.
- "$\subsetneq$" — For sets S and T, "$S \subsetneq T$" means the same thing as "$S \subset T$". Sometimes "$S \subsetneq T$" is used instead of "$S \subset T$" because people get "$\subseteq$" and "$\subset$" mixed up.
- "$\supseteq$" — For sets S and T. "$S \supseteq T$" means the same thing as "$T \subseteq S$".
- "$\supset$" — For sets S and T, "$S \supset T$" means the same thing as "$T \subset S$" (which means the same thing as $T \subsetneq S$", as noted above).
- "$\supsetneq$" — For sets S and T, "$S \supsetneq T$" means the same thing as "$S \supset T$" (or "$T \subset S$" or "$T \subsetneq S$", as noted above).
- "$=$" — For sets S and T, "$A = B$" means that, for every object a, $a \in A$ if and only if $a \in B$.

Suppose, now, that A and B are sets.

- The ***union*** of A and B, $A \cup B$, is the set of objects a such that either $a \in A$ or $a \in B$, or both.
- The ***intersection*** of A and B, $A \cap B$, is the set of objects a such that $a \in A$ *and* $a \in B$.
- $A \setminus B$ is the set of objects a such that $a \in A$ and $a \notin B$.

Sequences

A ***sequence*** of objects is a list of objects in some order. There does not seem to be a standard way to represent these. For example, the sequence with "length" five, consisting of the integers 1, 3, 1 (again), 2 and 4 will generally be shown by listing the entries of the sequence, separated by commas:

$$1, 3, 1, 2, 4$$

However, a finite sequence is sometimes shown as enclosed by parenthesis,

$$(1, 3, 1, 2, 4)$$

or angle brackets,

$$\langle 1, 3, 1, 2, 4 \rangle$$

Finite sequences are sometimes called ***tuples*** and, if k is a positive integer, then a finite sequence with length k is sometimes called a ***k-tuple***. Furthermore, 2-tuples are also called ***ordered pairs***.

Cartesian Products

If A and B then the ***Cartesian product*** $A \times B$ is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. More generally, if k is a positive integer and $A_1, A_2, \ldots, A_k$ are sets, then $A_1 \times A_2 \times \cdots \times A_k$ is the set of k-tuples $(a_1, a_2, \ldots, a_k)$ such that $a_i \in A_i$ for every integer i such that $1 \leq i \leq k$.

Functions and Relations

If A and B are sets then a ***function*** from A to B is an assignment of exactly one object $b \in B$ to every object $a \in A$. Functions from A to B are sometimes called ***total functions*** from A to B; a ***partial function*** from A to B is an assignment of ***at most*** one value $b \in B$ to every value $a \in A$.

The notation

$$f : A \to B$$

represents the fact that f is a partial function from A to B. For any such function f, if $a \in A$ and there exists an object $b \in B$ such that f associates b with a, then we say that f is ***defined*** at a and we write the value associated with a, by f, as $f(a)$. On the other hand, if there is no value $b \in B$ associated with a by f, then we say that f is ***undefined*** at a (or, sometimes, say that "$f(a)$ is undefined").

If $f : A \to B$ then A is called the ***domain*** of the function f, and B is called the ***range*** of f.

Suppose, once again, that $f : A \to B$.

- The function f is called ***injective***, or **1-1**, if it satisfies the following property: For every object $b \in B$, there exists ***at most*** one object $a \in A$ such that $f(a) = b$ — so that, for all objects $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$ then $a_1 = a_2$.
- The function f is called ***surjective***, or ***onto***, if f satisfies the following property: For every object $b \in B$, there exists *at least* one object $a \in A$ such that $f(a) = b$.
- A function $f : A \to B$ is called ***bijective*** if f is both injective and surjective — so that, for every object $b \in B$, there exists ***exactly one*** object $a \in A$ such that $f(a) = b$. A function $f : A \to B$, that is bijective, is called a ***bijection*** from A to B.

When $f : A_1 \times A_2 \times \cdots \times A_k \to B$ for some positive integer k and for sets $A_1, A_2, \ldots, A_k$ and B, we call f a ***k-ary*** function. This is called a "unary" function when $k = 1$ and it is called a "binary" function when $k = 2$.

A ***predicate***, or ***property***, is a function whose range is $\{\text{true}, \text{false}\}$.[1] A property whose domain is the set of k-tuples $A \times A \times \cdots \times A$, for some positive integer k and set A, is called a ***relation*** (or a ***k-ary relation***, or a ***k-ary relation on A***).

Infix notation is used when defining and using various familiar binary functions and relations. For the example, while the "integer addition function" would be represented as a total function $+ : \mathbb{Z} \times Z \to \mathbb{Z}$, based on what is given above, we generate wrote "$a + b$" instead of "$+(a, b)$", as what you get when applying this function to the pair of integers a and b.

References

[1] Susanna S. Epp. *Discrete Mathematics with Applications*. Brooks Cole, fifth edition, 2019.

[2] Kenneth H. Rosen. *Discrete Mathematics and Its Applications*. McGraw-Hill Education, eighth edition, 2018.

[3] Michael Sipser. *Introduction to the Theory of Computation*. CENGAGE Learing, third edition, 2013.

[1] The symbols "true" and "false" are sometimes replaced by `T` and `F`, respectively, or by 0 and 1, respectively.