Asymptotic Notation and Standard Functions

Solutions for a Suggested Exercise

1. For each of the following functions f and g, you were asked to use asymptotic notation to express the relationship between these functions as you can — assuming that k, ϵ and c are constants such that $k \geq 1$, $\epsilon > 0$, and $c > 1$.

   (a) $f(n) = \log_2^k n$ and $g(n) = n^\epsilon$.
   Solution: $\log_2^k n \in o(n^\epsilon)$.
   Justification: It can be shown, by induction on k, that if $k \geq 0$ then

   $$\lim_{n \to +\infty} \frac{\log_2^k n}{n^\epsilon} = 0.$$

   l'Hôpital's rule is used in the inductive step.

   (b) $f(n) = n^k$ and $g(n) = c^n$.
   Solution: $n^k \in o(c^n)$.
   Justification: See the information about the relationship between polynomial functions and exponential functions included in Lecture #6.

   (c) $f(n) = \sqrt{n}$ and $g(n) = n^{\sin n}$ — assuming here, that (when computing $\sin n$) n is some number of *degrees* rather than *radians*.
   Solution: There is ***no*** relationship between these functions that can be expressed using the kinds of asymptotic notation introduced in this course!
   Justification: Notice that — when n is a value given in degrees — $\sin(n)$ has value 1 infinitely often (and for arbitrarily large n). Thus $g(n) = n$ infinitely often, so that $g(n) = f(n)^2$ infinitely often (and, again, for arbitrarily large n). If we restricted attention to the values of n such that this is the case then it would seem that $f(n) \in o(g(n))$.
   On the other hand, $\sin(n)$ has value -1 infinitely often (and for arbitrarily large n). If we restricted attention to the values of n that this is the case then it would seem like $f(n)$ was *increasing* with n while $g(n)$ was *decreasing* with n, and approaching zero, so that $f(n) \in \omega(g(n))$ instead.
   Indeed, an examination of the relationships between f and g included in the definitions for $O(g)$, $\Omega(g)$, $\Theta(g)$, $o(g)$ and $\omega(g)$ confirms that none of them are satisfied.

(d) $f(n) = 2^n$ and $g(n) = 2^{n/2}$
Solution: $2^n \in \omega(2^{n/2})$.
Justification: See the information about the relationship between exponential functions included in Lecture #6.

(e) $f(n) = n^{\log_2 c}$ and $g(n) = c^{\log_2 n}$
Solution: $f \in \Theta(g)$.
Justification

$$f(n) = n^{\log_2 c} = 2^{(\log_2 n)\cdot(\log_2 c)} = c^{\log_2 n} = g(n)$$

so that, in fact, $f(n) = g(n)$ — these are just two different ways to write the same function!

2. Let $f(n) = 3n^3 + 2n + 1$ and let $g(n) = n^3$.

(a) You were asked to use the *definition* of $O(g)$ to prove that $f \in O(g)$.
Solution:
Claim: $3n^3 + 2n + 1 \in O(n^3)$.
Proof: It follows by the definition of "$O(n^3)$" that it suffices to show that *there exist* constants $c > 0$ and $N_0 \geq 0$ such that, *for all* n in the domain of f and g such that $n \geq N_0$, $n^3 + 2n + 1 \leq c \cdot n^3$.
Let $c = 6$ and N_0. Then c and N_0 are certainly constants such that $c > 0$ and $N_0 \geq 0$.
Now let x be an arbitrarily chosen real number[1] such that $x \geq N_0 = 1$. Then

$$\begin{aligned} 3n^3 + 2n + 1 &\leq 3n^3 + 2n^3 + n^3 \qquad \text{(since } n \leq n^3 \text{ and } 1 \leq n^3 \text{ if } n \geq 1\text{)} \\ &= 6n^3 \\ &= c \cdot n^3, \end{aligned}$$

as required.
Now, since x was arbitrarily chosen such that $x \leq N_0 = 1$, it follows that $n^3 + 3n + 1 \leq c \cdot n^3$ for *all* n in the domain of f such that $n \geq N_0$.
Thus, *there exist* constants $c > 0$ and $N_0 \geq 0$ such that $n^3 + 3n + 1 \leq c \cdot n^3$ for *all* n in the domain of f such that $n \geq N_0$.
It follows by the definition of "$O(n^3)$" that $n^3 + 3n + 1 \in O(n^3)$. □

(b) You were asked to use a *limit test* to prove that $f \in O(g)$.
Solution: Note that

$$\lim_{n\to+\infty} \frac{n^3 + 3 + 1}{n^3} = \lim_{n\to+\infty} \left(1 + \frac{3}{n^2} + \frac{1}{n^3}\right) = 3.$$

[1] or natural number: The argument is the same, either way.

Since the limit of this ratio exists and is a constant — not equal to $+\infty$ — it follows by the "Limit Test for $O(g)$" that $n^3 + n + 1 \in O(n^3)$.

(c) Now that you have computed the limit needed to answer the previous part of this question, you were asked to consider the other "limit tests" for asymptotic notation. What *else* can concluded, about the relationship between f and g, based on the limit that you have computed?

Solution: Since the above limit is a *positive* constant — not equal to 0 — it follows by the "Limit Test for $\Omega(g)$" that $n^3 + 3n + 1 \in \Omega(n^3)$ as well.

This implies that $n^3 + 3n + 1 \in \Theta(n^3)$.

Since the limit is neither equal to 0 or $+\infty$, the "Limit Test for $o(g)$" and the "Limit Test for $\omega(g)$" can be applied to conclude that $n^3+3n+1 \notin o(n^3)$ and $n^3+3n+1 \notin \omega(n^3)$.