Review of Proofs and Mathematical Induction
Solution for Suggested Exercise

For this exercise, you were asked to consider **claim:** For every integer n such that $n \geq 2$,

$$\prod_{i=2}^{n} \left(1 - \tfrac{1}{i}\right) = \tfrac{1}{n}.$$

You were asked to suppose that you want to write a ***proof*** of this claim, using mathematical induction — specifically, induction on n, using the standard form of mathematical induction.

(a) You were first asked to write down the ***result*** that you need to prove in the basis.

Solution: Since the basis concerns the case that $n = 2$, the result you need to establish is that

$$\prod_{i=2}^{2} \left(1 - \tfrac{1}{i}\right) = \tfrac{1}{2}.$$

(b) You were next asked to write down the ***Inductive Hypothesis*** — which has something to do with the case that $n = k$ where k is an integer such that $k \geq 2$, as described in the exercise.

Solution: The ***Inductive Hypothesis*** is as follows:

$$\prod_{i=2}^{k} \left(1 - \tfrac{1}{i}\right) = \tfrac{1}{k}.$$

(c) You were then asked to write down the ***Inductive Claim*** — which has something to do with the case that $n = k + 1$, for k as above.

Solution: The ***Inductive Claim*** is as follows:

$$\prod_{i=2}^{k+1} \left(1 - \tfrac{1}{i}\right) = \tfrac{1}{k+1}.$$

(d) Finally, you were asked to fill in the details in order to provide a complete proof for this claim.

Solution: A proof for this claim is as follows.

Claim: For every integer n such that $n \geq 2$,

$$\prod_{i=2}^{n} \left(1 - \tfrac{1}{i}\right) = \tfrac{1}{n}.$$

Proof: By mathematical induction on n. The standard form of mathematical induction will be used.

Basis: If $n = 2$ then

$$\prod_{i=2}^{n} \left(1 - \tfrac{1}{i}\right) = \prod_{i=2}^{2} \left(1 - \tfrac{1}{i}\right) = 1 - \tfrac{1}{2} = \tfrac{1}{2} = \tfrac{1}{n},$$

as needed to establish the claim in this case.

Inductive Step: Let k be an integer such that $k \geq 2$. It is necessary and sufficient to use the following

Inductive Hypothesis:

$$\prod_{i=2}^{k} \left(1 - \tfrac{1}{i}\right) = \tfrac{1}{k}.$$

to prove the following

Inductive Claim:

$$\prod_{i=2}^{k+1} \left(1 - \tfrac{1}{i}\right) = \tfrac{1}{k+1}.$$

With that noted: If k is as above then

$$\begin{aligned}
\prod_{i=2}^{k+1} \left(1 - \tfrac{1}{i}\right) &= \left(1 - \tfrac{1}{k+1}\right) \times \prod_{i=2}^{k} \left(1 - \tfrac{1}{i}\right) \\
&= \tfrac{k}{k+1} \times \prod_{i=2}^{k} \left(1 - \tfrac{1}{i}\right) \\
&= \tfrac{k}{k+1} \times \tfrac{1}{k} && \text{(by the Inductive Hypothesis)} \\
&= \tfrac{1}{k+1},
\end{aligned}$$

as needed to establish the Inductive Claim, as needed to complete the Inductive step and the proof of the claim. □

Here are some things to ***watch for and avoid:***

- I have set a question like this before. A surprisingly large number of students skip over parts (a)–(c) of the question and only answer part (d).

  Markers are not, in fact, expected or obliged to hunt through an answer for (d) in search for answers to parts (a), (b), and (c). Consequently, even if the answer for (d) is perfect, or close to it, the mark assigned for the question can be D- of F-level, rather than A-level.

- Students can also take too many shortcuts when writing — leaving out most of a required proof. Sometimes, even if a marker *wanted* to try to give credit for parts (a), (b), and (c), above, this would not be possible, because the parts of the proof being asked for, above, never get written down.

  Markers do not read minds. They are also not supposed to assume that you know something until you prove that you do not: It is the other way around.

- Please see the assigned reading for a discussion of additional problems (generally writing-related) that should be avoided when solving this kind of problem.