CPSC 641: Analytical Solution for Assignment 2

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The Markov chain is reasonably straightforward, but too hard to draw in LaTeX. I have the diagram if you need to see it. The key insight is using $\lambda/2$ as the rate on the two different transitions out of state 0 (empty), since choosing between the two barbers is done at random, in an equi-likely fashion (they are twins!).

In terms of notation, p_i represents the probability of having i customers in the system, with p_{1F} and p_{1G} used to keep track of whether a single customer is with either Faruz or Gemal. Let their respective service rates be μ_F (Faruz) and μ_G (Gemal).

The global flow balance equations (state view) are:

$$\lambda p_0 = \mu_F p_{1F} + \mu_G p_{1G}$$

$$(\lambda + \mu_F)p_{1F} = \frac{\lambda}{2}p_0 + \mu_G p_2$$

$$(\lambda + \mu_G)p_{1G} = \frac{\lambda}{2}p_0 + \mu_F p_2$$

$$(\lambda + \mu_F + \mu_G)p_2 = \lambda(p_{1F} + p_{1G}) + (\mu_F + \mu_G)p_3$$

$$(\lambda + \mu_F + \mu_G)p_3 = \lambda p_2 + (\mu_F + \mu_G)p_4$$

$$(\lambda + \mu_F + \mu_G)p_4 = \lambda p_3 + (\mu_F + \mu_G)p_5$$

$$\lambda p_4 = (\mu_F + \mu_G)p_5$$

The equations for the queue can of course be generalized for larger k.

It can also be shown that local flow balance (cut view) holds, in which case:

$$\frac{\lambda}{2}p_0 = \mu_F p_{1F}$$

$$\frac{\lambda}{2}p_0 = \mu_G p_{1G}$$

$$\lambda p_{1F} = \mu_G p_2$$

$$\lambda p_{1G} = \mu_F p_2$$

$$\lambda p_2 = (\mu_F + \mu_G)p_3$$

$$\lambda p_3 = (\mu_F + \mu_G)p_4$$

$$\lambda p_4 = (\mu_F + \mu_G)p_5$$

With a bit of algebra, one can show that:

$$p_{1F} = \frac{\lambda}{2\mu_F} p_0$$

$$p_{1G} = \frac{\lambda}{2\mu_G} p_0$$

$$p_2 = \frac{\lambda^2}{2\mu_F \mu_G} p_0$$

$$p_3 = \frac{\lambda^3}{2\mu_F \mu_G (\mu_F + \mu_G)} p_0 = \rho p_2$$

$$p_4 = \frac{\lambda^4}{2\mu_F \mu_G (\mu_F + \mu_G)^2} p_0 = \rho p_3 = \rho^2 p_2$$

$$p_5 = \frac{\lambda^5}{2\mu_F \mu_G (\mu_F + \mu_G)^3} p_0 = \rho p_4 = \rho^3 p_2$$

where $\rho = \frac{\lambda}{\mu_F + \mu_G}$ is the offered load.

After a LOT of algebraic manipulation (and some strong coffee!), one can show that:

$$p_0 = \frac{2\mu_F \mu_G (\mu_F + \mu_G)^3}{\lambda^5 + \lambda^4(\mu_F + \mu_G) + \lambda^3(\mu_F + \mu_G)^2 + \lambda^2(\mu_F + \mu_G)^3 + \lambda(\mu_F + \mu_G)^4 + 2\mu_F \mu_G (\mu_F + \mu_G)^3}$$

Note the beautiful structure of the equations, and the paired roles of μ_F and μ_G.

I wrote a simple C program with the foregoing equations. For $\lambda = 6$, $\mu_F = 5$, and $\mu_G = 3$, my program indicates an average utilization of 0.83, with Faruz's chair occupied 0.66 of the time, and Gemal's chair occupied 0.73 of the time. The average system occupancy is 2.03 customers, and the proportion of lost customers is 8.6%. This agrees very well with my simulation results.