Substitution Systems

More Explorations into Simple Programs

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Substitution Systems versus Cellular Automata

Cellular automata work on a **fixed array of cells**, whether in one, two, three, or N dimensions.

This means, while the colours or states of the cells can be updated following a wide range of different possible rules, the underlying number and organization of cells always stays the same.

In substitution systems, on the other hand, the number of elements can change.

Substitution Systems

Let us have a look at some variants of substitution systems and their evolutionary dynamics.

Typical Simple Substitution System

In the typical case, we have a sequence of elements—each coloured say **black** or **white**—and at each step each one of these elements is replaced by a new block of elements.

■ Implementing a Simple Substitution System

So, let's start to implement the following rules:

```
ruleSet = {1 → {1, 0}, 0 → {0, 1}};
```

We apply these rules to a "1" as the initial element, and apply the rule set several times:

```
1 /. ruleSet
```

```
{1, 0}
```

```
% /. ruleSet
```

```
{{1, 0}, {0, 1}}
```

```
% /. ruleSet
```

```
{{{1, 0}, {0, 1}}, {{0, 1}, {1, 0}}}
```

If we want to get rid of the inner brackets, we can flatten the expression:

```
Flatten[%]
```

```
{1, 0, 0, 1, 0, 1, 1, 0}
```

■ Automization of the Step-by-Step Rule Application

We can automize this step-by-step application of rules by using the *NestList* function. What does *NestList* do?

```
NestList[f, a, 3]
```

```
{a, f[a], f[f[a]], f[f[f[a]]]}
```

Now we could define *f* to perform the rule applications defined in *ruleSet*:

```
f[x_] := x /. ruleSet
```

```
NestList[f, {1}, 3]
```

```
{{1}, {{1, 0}}, {{{1, 0}, {0, 1}}}, {{{{1, 0}, {0, 1}}, {{0, 1}, {1, 0}}}}}
```

```
% // ColumnForm
```

```
{1}
{{1, 0}}
{{{1, 0}, {0, 1}}}
{{{{1, 0}, {0, 1}}, {{0, 1}, {1, 0}}}}
```

■ Flattening the Lists

With an additional *Flatten* command, we can get rid of all the inner brackets. The following definition for *f* overwrites the previous one:

```
f[x_] := Flatten[x /. ruleSet]
```

```
NestList[f, {1}, 3]
```

```
{{1}, {1, 0}, {1, 0, 0, 1}, {1, 0, 0, 1, 0, 1, 1, 0}}
```

```
% // ColumnForm
```

```
{1}
{1, 0}
{1, 0, 0, 1}
{1, 0, 0, 1, 0, 1, 1, 0}
```

■ Generalization to Automatic Substitution

This function *f* is, of course, tied to the definition of *ruleSet*, as we have assigned it above:

```
ruleSet
```

```
{1 → {1, 0}, 0 → {0, 1}}
```

In order to write a more general evolution function for a substitution system, we can proceed as follows. We implement a function that takes three arguments:

(1) the rule set (list of substitutions)

(2) an initial expression

(3) the number of iterations (how many times to apply the rules)

Here is such a function, which we call *SSEvolveList*:

```
SSEvolveList[rule_, init_List, t_Integer] := NestList[Flatten[# /. rule] &, init, t]
```

■ Example 1: Checking out the New Substitution System Function

Let us check, whether it works:

```
SSEvolveList[{1 → {1, 0}, 0 → {0, 1}}, {1}, 8]
```

```
{{1}, {1, 0}, {1, 0, 0, 1}, {1, 0, 0, 1, 0, 1, 1, 0},
 {1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1}, {1, 0, 0, 1, 0, 1, 1, 0,
  0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0},
 {1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0,
  0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0,
  1, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1},
 {1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1,
  0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0,
  1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0,
  1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1,
  0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0},
 {1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1,
  0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0,
  1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0,
  1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1,
  0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0,
  1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1,
  0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1,
  0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0,
  0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0,
  1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1}}
```

```
% // ColumnForm
```

```
{1}
{1, 0}
{1, 0, 0, 1}
{1, 0, 0, 1, 0, 1, 1, 0}
{1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1}
{1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0,
{1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0,
{1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0,
{1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0,
```

The visualization of these evolving structures will need some work. Here is one way to visualize it.

```
showRectangleEvolution[%%];
```

■ Example 2

For now, let's have a look at another simple substitution system, this time one that does not grow as fast:

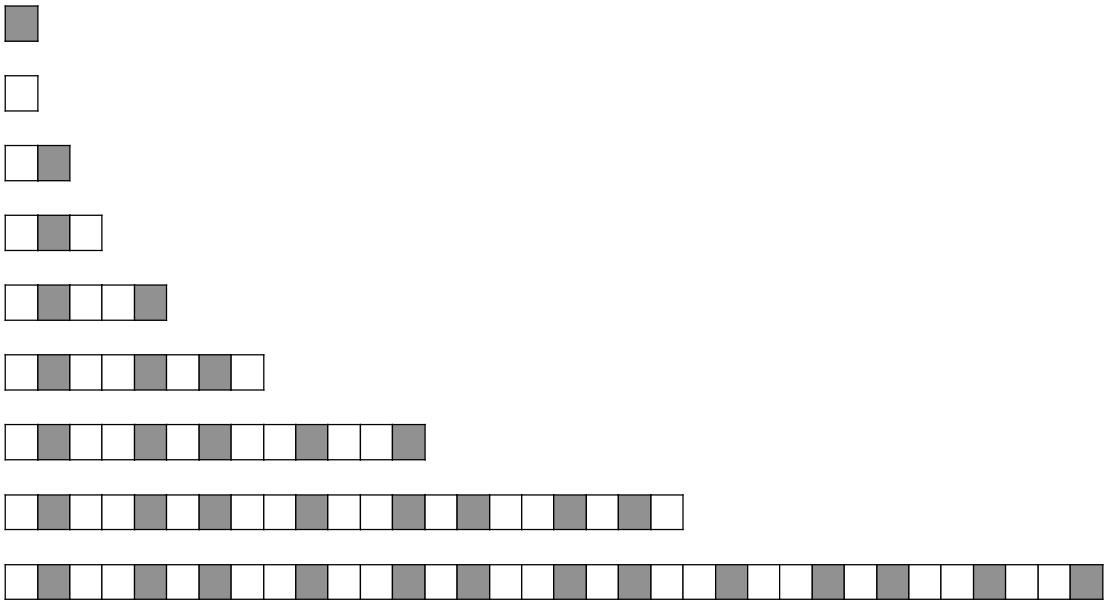
```
SSEvolveList[{1 → 0, 0 → {0, 1}}, {1}, 8]
```

```
{{1}, {0}, {0, 1}, {0, 1, 0}, {0, 1, 0, 0, 1},
 {0, 1, 0, 0, 1, 0, 1, 0}, {0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1},
 {0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0}, {0, 1, 0, 0, 1, 0,
  1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1}}
```

```
% // ColumnForm
```

```
{1}
{0}
{0, 1}
{0, 1, 0}
{0, 1, 0, 0, 1}
{0, 1, 0, 0, 1, 0, 1, 0}
{0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1}
{0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0}
{0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0,
```

```
showRectangleEvolution[%%];
```



In these simple cases, we are dealing with **contextfree** rules. That is, each element of a particular colour is replaced by a fixed block of new elements, independent of the colours of the neighbouring elements.

■ Example 3

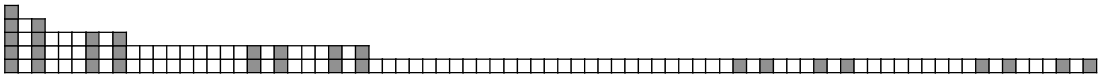
```
example[3] = SSEvolveList[{1 → {1, 0, 1}, 0 → {0, 0, 0}}, {1}, 5]
```

```
{{1}, {1, 0, 1}, {1, 0, 1, 0, 0, 0, 1, 0, 1},
 {1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1},
 {1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
  0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1}}
```

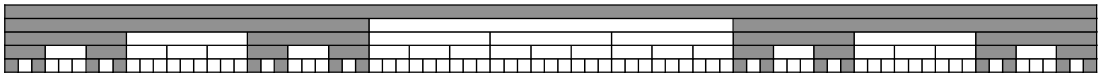
```
% // ColumnForm
```

```
{1}
{1, 0, 1}
{1, 0, 1, 0, 0, 0, 1, 0, 1}
{1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1}
{1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0,
```

```
showRectangleEvolution[%%];
```



```
showRectangleEvolutionStrech[%%%];
```



■ Example 4

NKS, *p. 84 (a)*

```
example[4] = SSEvolveList[{1 → {0, 0}, 0 → {0, 1}}, {1}, 7]
```

```
{{1}, {0, 0}, {0, 1, 0, 1}, {0, 1, 0, 0, 0, 1, 0, 0},
 {0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1}, {0, 1, 0, 0, 0, 1, 0, 1,
  0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0},
 {0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1,
  0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0,
  0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1},
 {0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0,
  1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0,
  0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0,
  1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0,
  1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0}}
```

```
% // ColumnForm
```

```
{1}
{0, 0}
{0, 1, 0, 1}
{0, 1, 0, 0, 0, 1, 0, 0}
{0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1}
{0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0,
{0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0,
{0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0,
```

```
showRectangleEvolution[%%];
```

```
showRectangleEvolutionStrech[%%%];
```

■ Example 5

NKS, *p. 84 (b)*

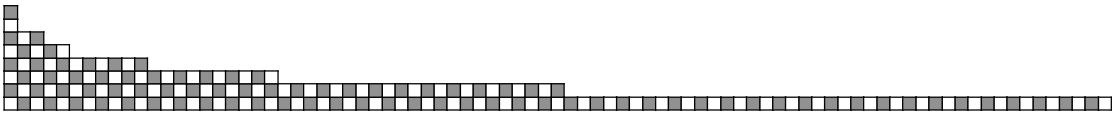
```
example[5] = SSEvolveList[{1 → 0, 0 → {1, 0, 1}}, {1}, 7]
```

```
{{1}, {0}, {1, 0, 1}, {0, 1, 0, 1, 0}, {1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1},
 {0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0},
 {1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1,
  0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1},
 {0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1,
  0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0,
  1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1,
  0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0}}
```

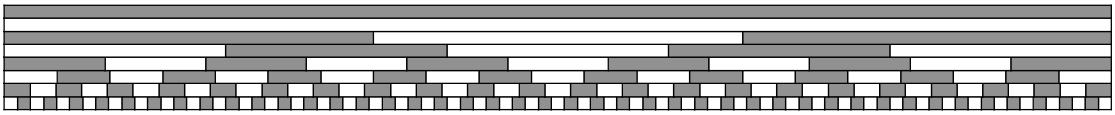
```
% // ColumnForm
```

```
{1}
{0}
{1, 0, 1}
{0, 1, 0, 1, 0}
{1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1}
{0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0}
{1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1,
{0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0,
```

```
showRectangleEvolution[%%];
```



```
showRectangleEvolutionStrech[%%%];
```



■ Example 6

NKS, *p. 84 (c)*

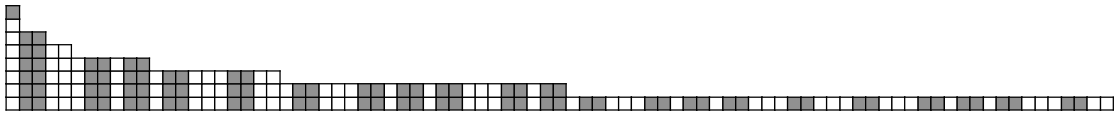
```
example[6] = SSEvolveList[{1 → 0, 0 → {0, 1, 1}}, {1}, 7]
```

```
{{1}, {0}, {0, 1, 1}, {0, 1, 1, 0, 0}, {0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1},
 {0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0},
 {0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0,
  0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1},
 {0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0,
  1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1,
  0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0,
  0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0}}
```

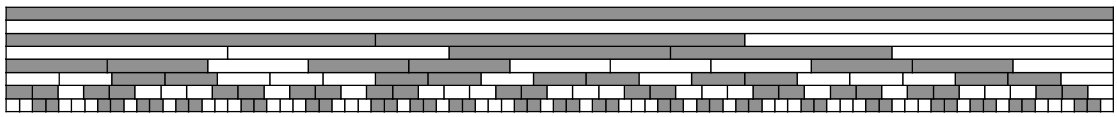
```
% // ColumnForm
```

```
{1}
{0}
{0, 1, 1}
{0, 1, 1, 0, 0}
{0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1}
{0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0}
{0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1,
{0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1,
```

```
showRectangleEvolution[%%];
```



```
showRectangleEvolutionStrech[%%%];
```



■ Example 7

NKS, *p. 84 (d)*

```
example[7] = SSEvolveList[{1 → 0, 0 → {0, 1, 0}}, {1}, 7]
```

```
{{1}, {0}, {0, 1, 0}, {0, 1, 0, 0, 0, 1, 0},
 {0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0},
 {0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0,
  0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0},
 {0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0,
  1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1,
  0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0,
  0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0},
 {0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1,
  0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1,
  0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0,
  1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0,
  1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0,
  1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0,
  0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1,
  0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0,
  1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0}}
```

```
% // ColumnForm
```

```
{1}
{0}
{0, 1, 0}
{0, 1, 0, 0, 0, 1, 0}
{0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0}
{0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0,
{0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0,
{0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0,
```

```
showRectangleEvolution[%%];
```

```
showRectangleEvolutionStrech[%%%];
```

■ Example 8

NKS, *p. 84 (e)*

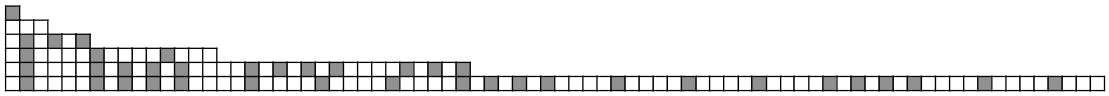
```
example[8] = SSEvolveList[{1 → {0, 0, 0}, 0 → {0, 1}}, {1}, 5]
```

```
{{1}, {0, 0, 0}, {0, 1, 0, 1, 0, 1},
 {0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 1, 0, 1,
  0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1},
 {0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0,
  0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0,
  0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0}}
```

```
% // ColumnForm
```

```
{1}
{0, 0, 0}
{0, 1, 0, 1, 0, 1}
{0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0}
{0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1,
{0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0,
```

```
showRectangleEvolution[%%];
```



```
showRectangleEvolutionStrech[%%%];
```

■ Example 9

NKS, *p. 84 (f)*

```
example[9] = SSEvolveList[{1 → {0, 1, 0}, 0 → {0, 1}}, {1}, 5]
```

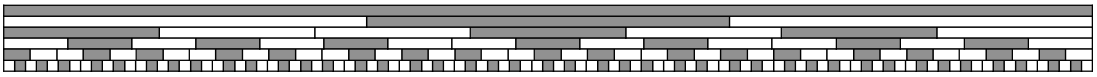
```
{{1}, {0, 1, 0}, {0, 1, 0, 1, 0, 0, 1},
 {0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0},
 {0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0,
  1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1},
 {0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0,
  1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1,
  0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0,
  0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0}}
```

```
% // ColumnForm
```

```
{1}
{0, 1, 0}
{0, 1, 0, 1, 0, 0, 1}
{0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0}
{0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0,
{0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0,
```

```
showRectangleEvolution[%%];
```

```
showRectangleEvolutionStrech[%%%];
```



■ Example 10

NKS, *p. 84 (g)*

```
example[10] = SSEvolveList[{1 → {1, 0, 0}, 0 → {0, 1}}, {1}, 5]
```

```
{{1}, {1, 0, 0}, {1, 0, 0, 0, 1, 0, 1},
 {1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0},
 {1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1,
  0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1},
 {1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0,
  0, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 0,
  1, 0, 1, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0,
  1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0}}
```

```
% // ColumnForm
```

```
{1}
{1, 0, 0}
{1, 0, 0, 0, 1, 0, 1}
{1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0}
{1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1,
{1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1,
```

```
showRectangleEvolution[%%];
```

```
showRectangleEvolutionStrech[%%%];
```

■ Example 11

NKS, *p. 84 (h)*

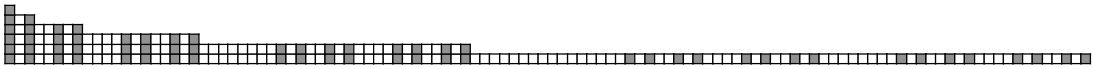
```
example[11] = SSEvolveList[{1 → {1, 0, 1}, 0 → {0, 0}}, {1}, 5]
```

```
{{1}, {1, 0, 1}, {1, 0, 1, 0, 0, 1, 0, 1},
 {1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1},
 {1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0,
  0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1},
 {1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0,
  1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1,
  0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1}}
```

```
% // ColumnForm
```

```
{1}
{1, 0, 1}
{1, 0, 1, 0, 0, 1, 0, 1}
{1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1}
{1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1,
{1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1,
```

```
showRectangleEvolution[%%];
```



```
showRectangleEvolutionStrech[%%%];
```

■ Example 12

NKS, *p. 84 (i)*

```
example[12] = SSEvolveList[{1 → {1, 0, 0}, 0 → {1, 0, 1}}, {1}, 5]
```

```
{{1}, {1, 0, 0}, {1, 0, 0, 1, 0, 1, 1, 0, 1},
 {1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0},
 {1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1,
  0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 1,
  0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1},
 {1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0,
  1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1,
  1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1,
  1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0,
  1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1,
  1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0,
  1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0,
  1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1,
  1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0}}
```

```
% // ColumnForm
```

```
{1}
{1, 0, 0}
{1, 0, 0, 1, 0, 1, 1, 0, 1}
{1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0}
{1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0,
{1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0,
```

```
showRectangleEvolution[%%];
```

```
showRectangleEvolutionStrech[%%%];
```

■ Example 13

NKS, *p. 84 (i)*

```
example[13] = SSEvolveList[{1 → {0, 0, 0}, 0 → {1, 0, 1}}, {1}, 5]
```

```
{{1}, {0, 0, 0}, {1, 0, 1, 1, 0, 1, 1, 0, 1},
 {0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0},
 {1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1,
  0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1,
  0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1},
 {0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0,
  1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1,
  0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0,
  0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0,
  1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1,
  0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0,
  0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0,
  1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1,
  0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0}}
```

```
% // ColumnForm
```

```
{1}
{0, 0, 0}
{1, 0, 1, 1, 0, 1, 1, 0, 1}
{0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0}
{1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0,
{0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0,
```

```
showRectangleEvolution[%%];
```

```
showRectangleEvolutionStrech[%%%];
```

Summary of the Simple Substitution System Examples

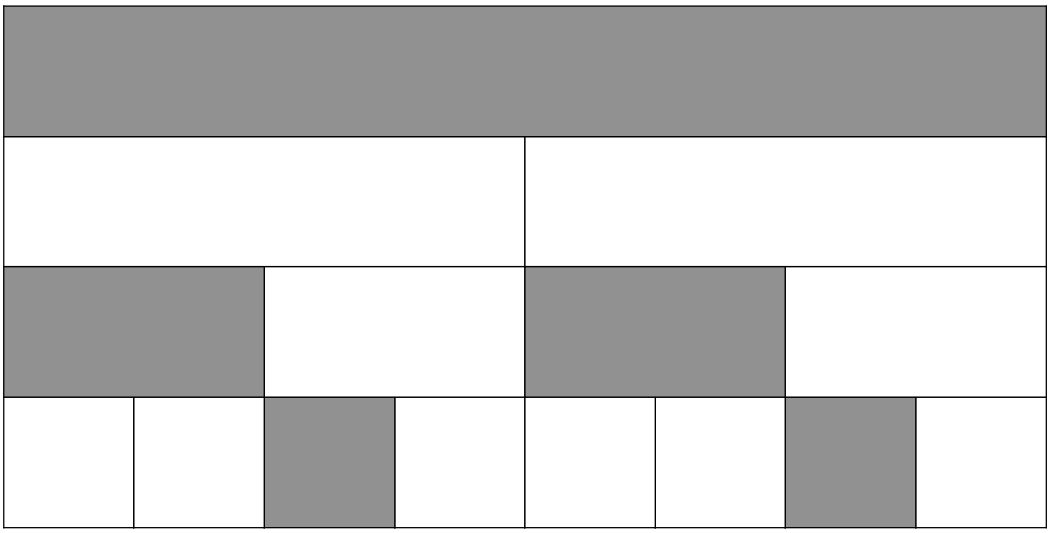
■ All the discussed rule systems

```
ruleSystems = {
   {1 → {0, 0}, 0 → {0, 1}}, (* Example 4 *)
   {1 → 0, 0 → {1, 0, 1}}, (* Example 5 *)
   {1 → 0, 0 → {0, 1, 1}}, (* Example 6 *)
   {1 → 0, 0 → {0, 1, 0}}, (* Example 7 *)
   {1 → {0, 0, 0}, 0 → {0, 1}}, (* Example 8 *)
   {1 → {0, 1, 0}, 0 → {0, 1}}, (* Example 9 *)
   {1 → {1, 0, 0}, 0 → {0, 1}}, (* Example 10 *)
   {1 → {1, 0, 1}, 0 → {0, 0}}, (* Example 11 *)
   {1 → {1, 0, 0}, 0 → {1, 0, 1}}, (* Example 12 *)
   {1 → {0, 0, 0}, 0 → {1, 0, 1}} (* Example 13 *)
  };
```

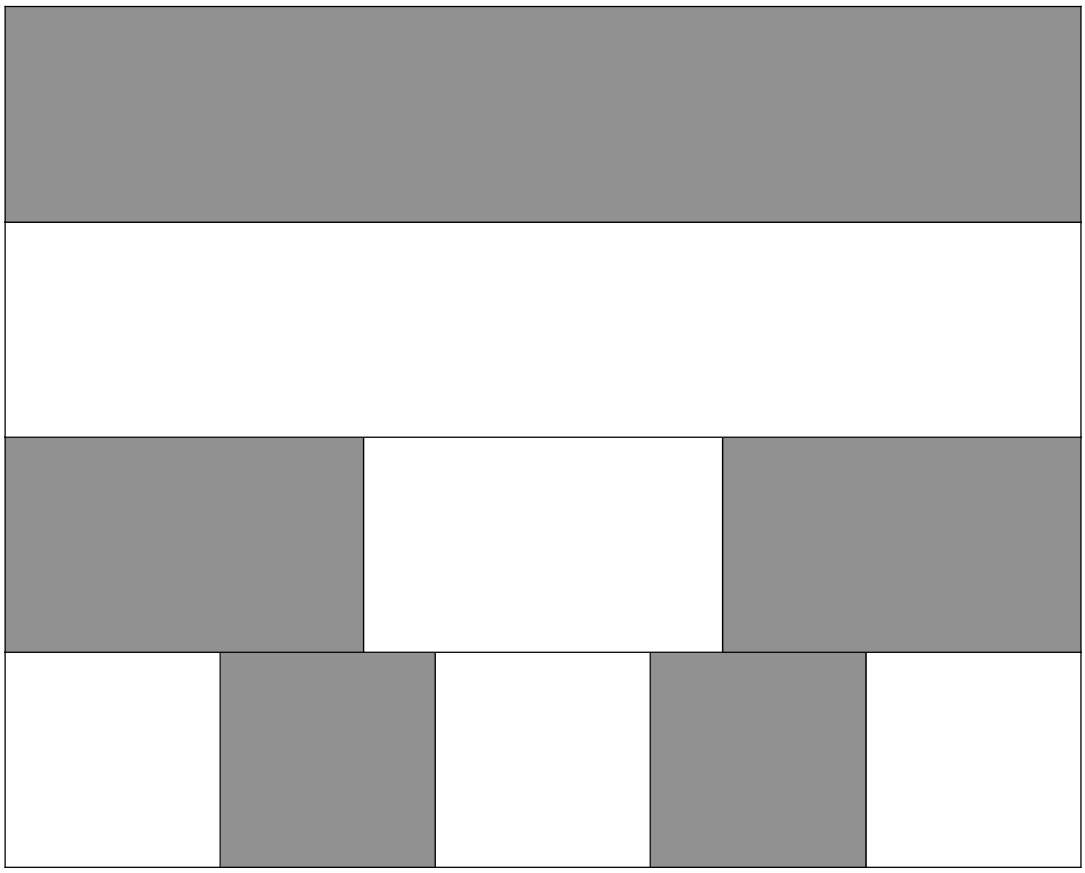
■ Visualize for 3 Iteration Steps

```
Map[showRectangleEvolutionStrechWithLabel[SSEvolveList[#, {1}, 3], #] &, ruleSystems];
```

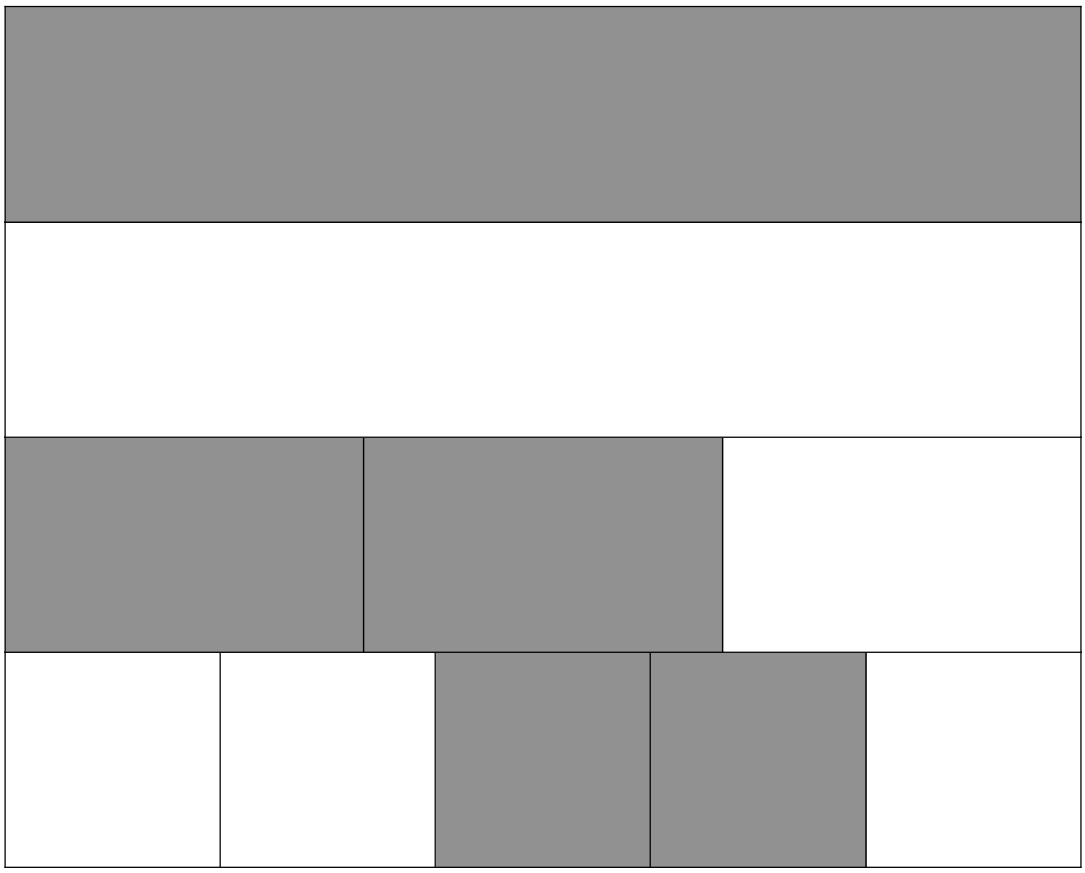
$\{1 \to \{0, 0\}, 0 \to \{0, 1\}\}$



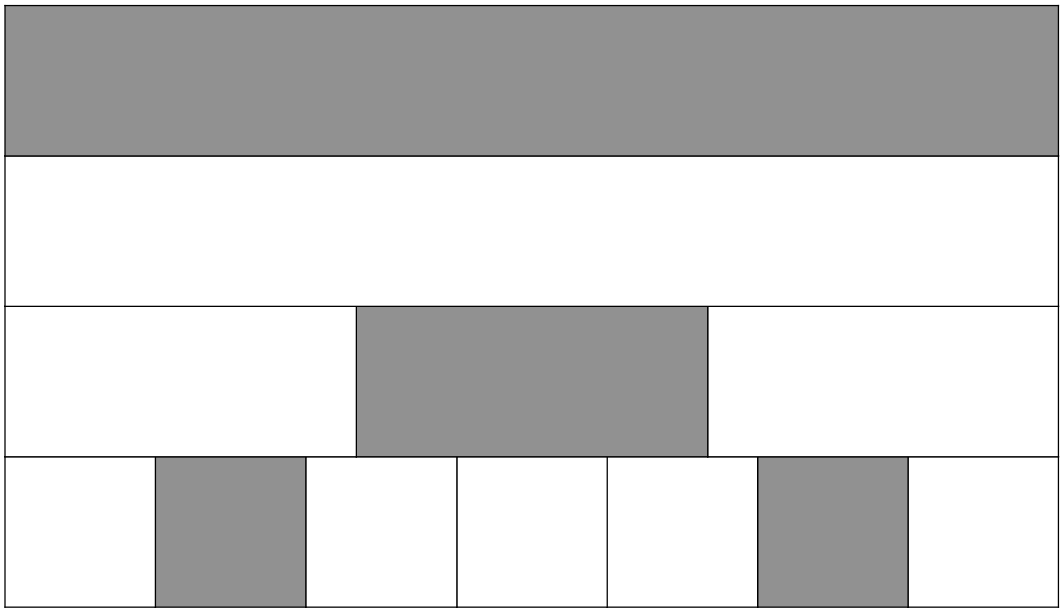
$$\{1 \to 0,\ 0 \to \{1,\ 0,\ 1\}\}$$



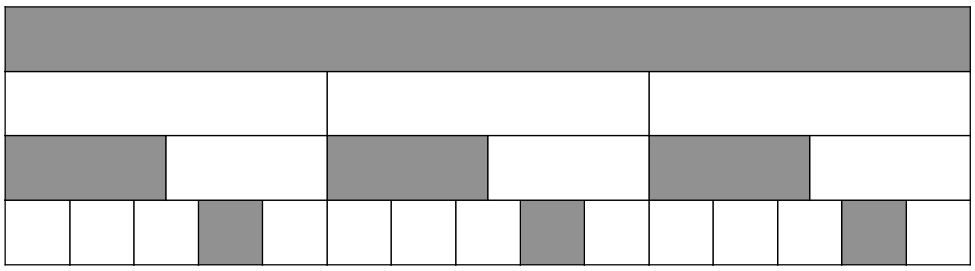
$$\{1 \to 0,\ 0 \to \{0,\ 1,\ 1\}\}$$



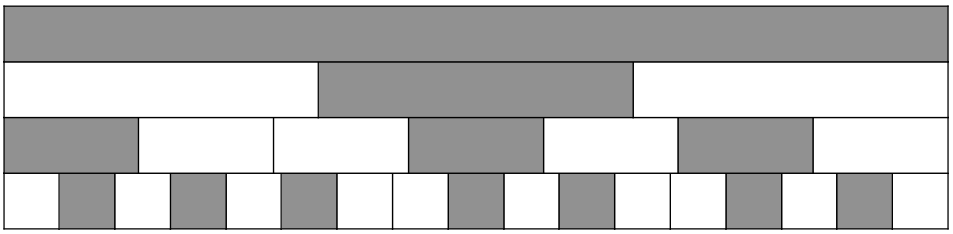
$$\{1 \to 0,\ 0 \to \{0,\ 1,\ 0\}\}$$



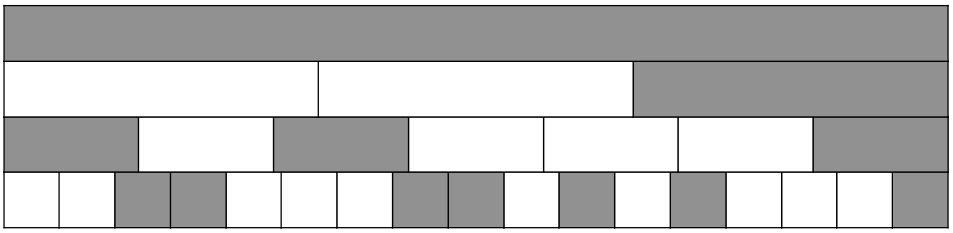
{1 → {0, 0, 0}, 0 → {0, 1}}



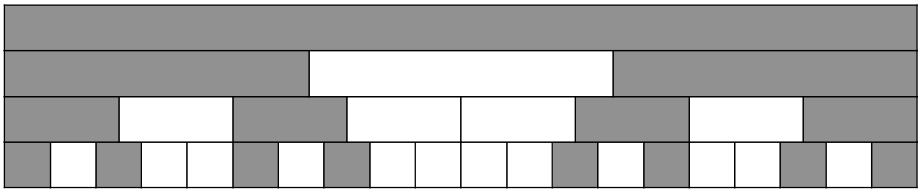
{1 → {0, 1, 0}, 0 → {0, 1}}



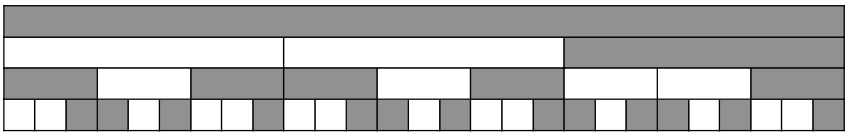
{1 → {1, 0, 0}, 0 → {0, 1}}



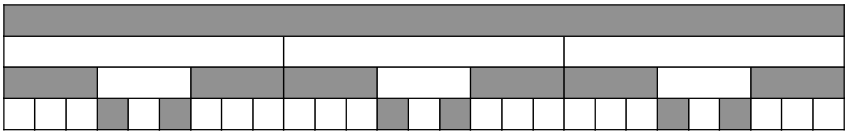
{1 → {1, 0, 1}, 0 → {0, 0}}



{1 → {1, 0, 0}, 0 → {1, 0, 1}}

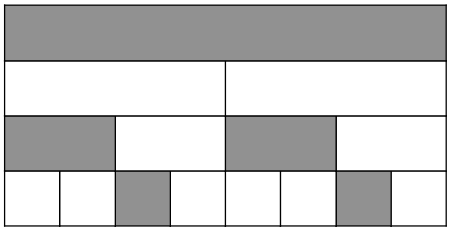


{1 → {0, 0, 0}, 0 → {1, 0, 1}}

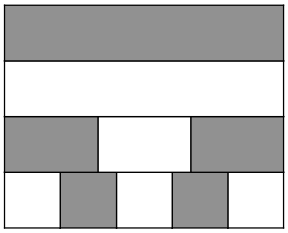


```
Show[GraphicsArray[Partition[%, 2]]];
```

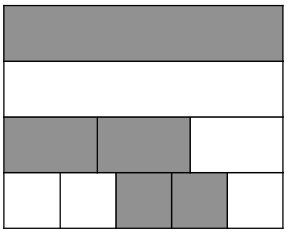
{1 → {0, 0}, 0 → {0, 1}}



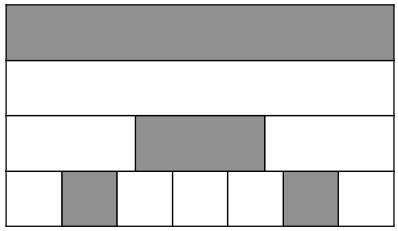
{1 → 0, 0 → {1, 0, 1}}



{1 → 0, 0 → {0, 1, 1}}

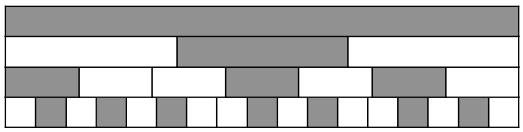


{1 → 0, 0 → {0, 1, 0}}



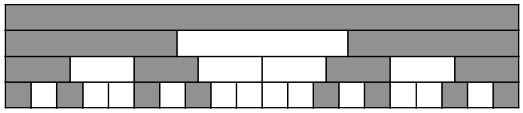
{1 → {0, 0, 0}, 0 → {0, 1}}

{1 → {0, 1, 0}, 0 → {0, 1}}

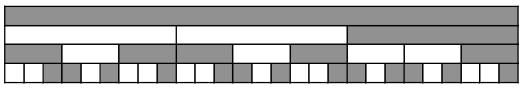


{1 → {1, 0, 0}, 0 → {0, 1}}

{1 → {1, 0, 1}, 0 → {0, 0}}



{1 → {1, 0, 0}, 0 → {1, 0, 1}}

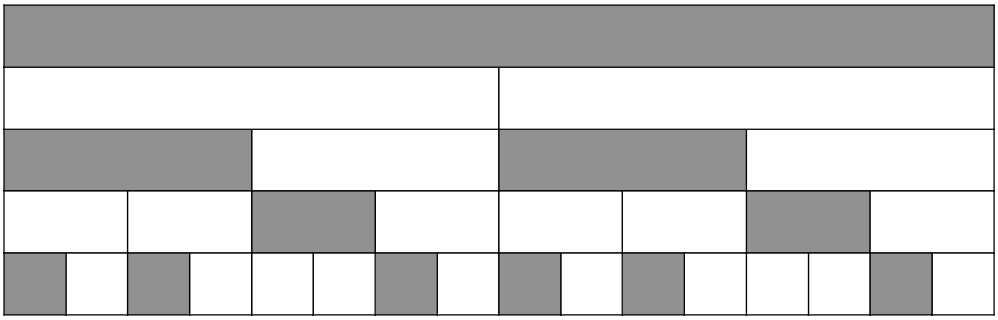


{1 → {0, 0, 0}, 0 → {1, 0, 1}}

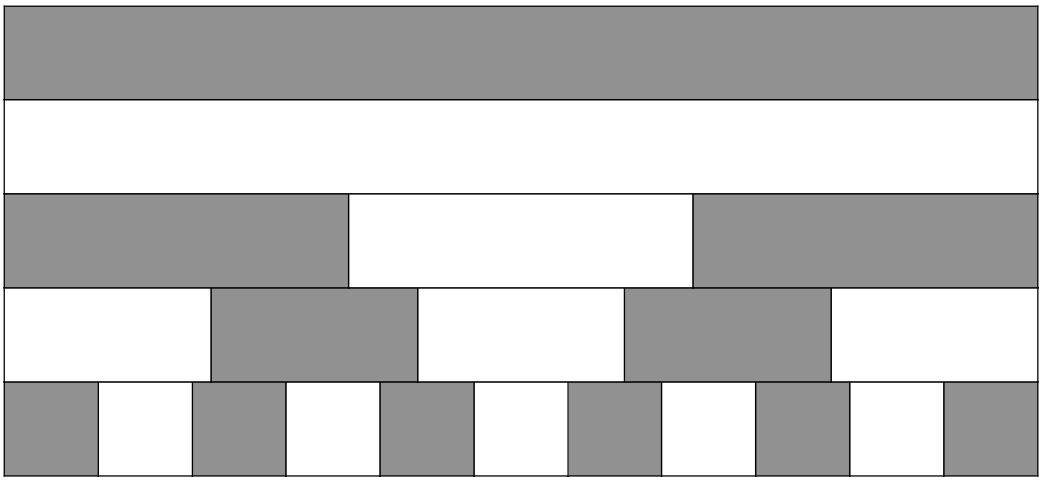
■ Visualize for 4 Iteration Steps

```
Map[showRectangleEvolutionStrechWithLabel[SSEvolveList[#, {1}, 4], #] &, ruleSystems];
```

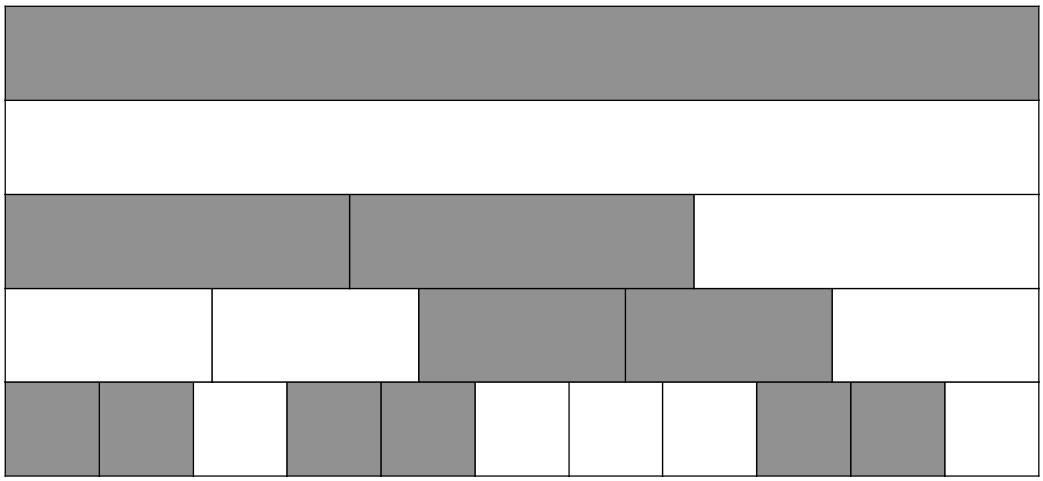
$\{1 \to \{0, 0\}, 0 \to \{0, 1\}\}$



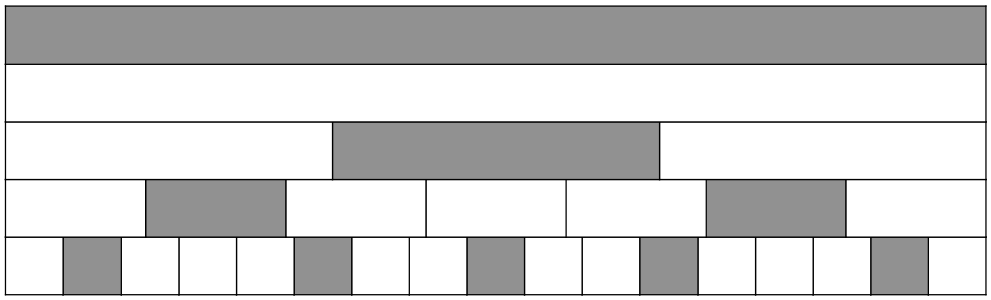
$\{1 \to 0, 0 \to \{1, 0, 1\}\}$



{1 → 0, 0 → {0, 1, 1}}

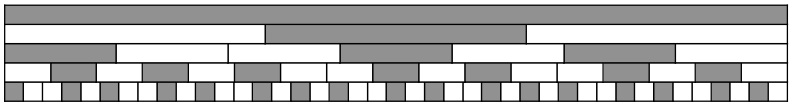


{1 → 0, 0 → {0, 1, 0}}



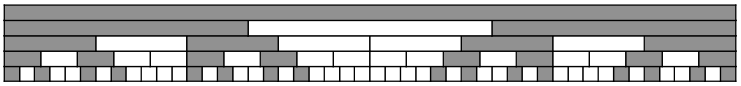
{1 → {0, 0, 0}, 0 → {0, 1}}

{1 → {0, 1, 0}, 0 → {0, 1}}

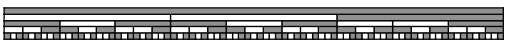


{1 → {1, 0, 0}, 0 → {0, 1}}

{1 → {1, 0, 1}, 0 → {0, 0}}



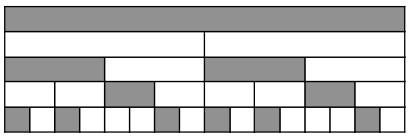
{1 → {1, 0, 0}, 0 → {1, 0, 1}}



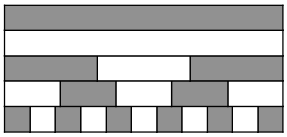
{1 → {0, 0, 0}, 0 → {1, 0, 1}}

```
Show[GraphicsArray[Partition[%, 2]]];
```

{1 → {0, 0}, 0 → {0, 1}}

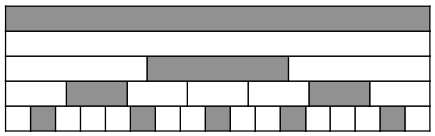


{1 → 0, 0 → {1, 0, 1}}



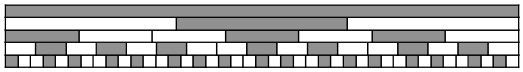
{1 → 0, 0 → {0, 1, 1}}

{1 → 0, 0 → {0, 1, 0}}

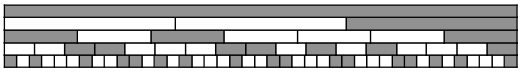


{1 → {0, 0, 0}, 0 → {0, 1}}

{1 → {0, 1, 0}, 0 → {0, 1}}

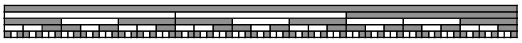


{1 → {1, 0, 0}, 0 → {0, 1}}

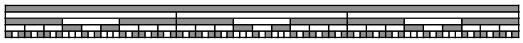


{1 → {1, 0, 1}, 0 → {0, 0}}

{1 → {1, 0, 0}, 0 → {1, 0, 1}}



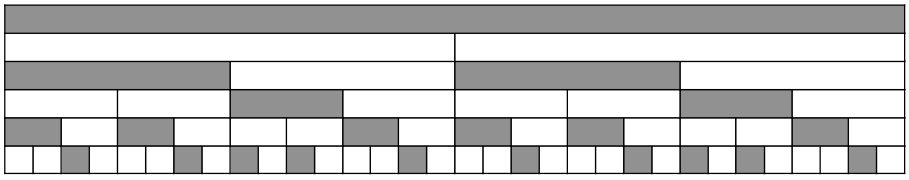
{1 → {0, 0, 0}, 0 → {1, 0, 1}}



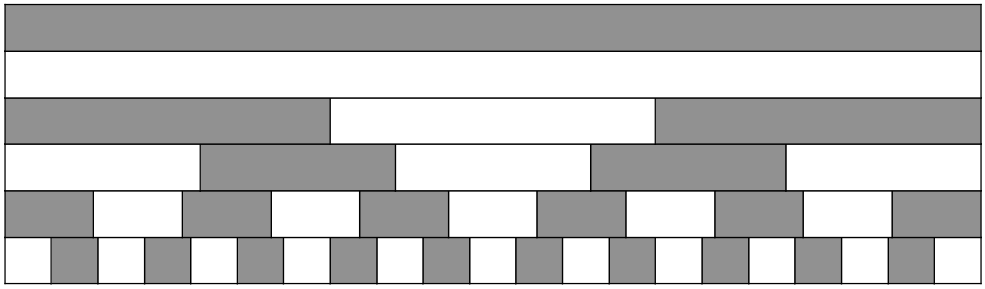
■ Visualize for 5 Iteration Steps

```
Map[showRectangleEvolutionStrechWithLabel[SSEvolveList[#, {1}, 5], #] &, ruleSystems];
```

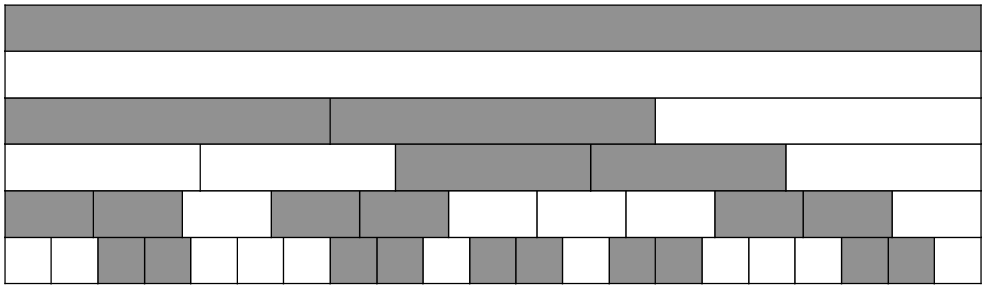
{1 → {0, 0}, 0 → {0, 1}}



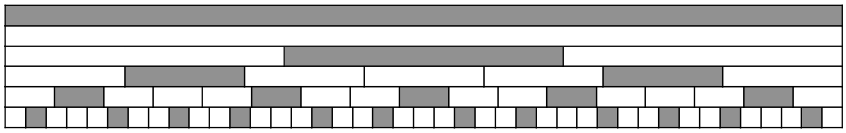
$$\{1 \to 0,\ 0 \to \{1,\ 0,\ 1\}\}$$



$$\{1 \to 0,\ 0 \to \{0,\ 1,\ 1\}\}$$



$$\{1 \to 0,\ 0 \to \{0,\ 1,\ 0\}\}$$



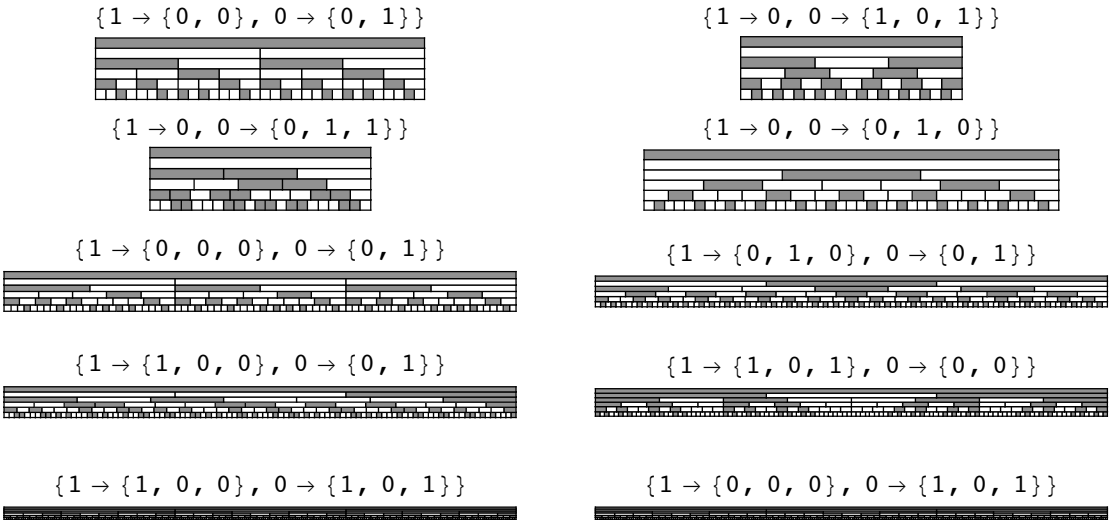
$$\{1 \to \{0,\ 0,\ 0\},\ 0 \to \{0,\ 1\}\}$$

$$\{1 \to \{0,\ 1,\ 0\},\ 0 \to \{0,\ 1\}\}$$

$$\{1 \to \{1,\ 0,\ 0\},\ 0 \to \{0,\ 1\}\}$$

$$\{1 \to \{1,\ 0,\ 1\},\ 0 \to \{0,\ 0\}\}$$

```
Show[GraphicsArray[Partition[%, 2]]];
```



More Complicated Substitution Systems

In order to get more complicated behaviour than simple nesting, we must consider substitution systems whose rules depend not only on the colour of a single element.

We now investigate **context-sensitive** substitution systems, whose rules depend on the colour of an element and on the colour of at least one of its neighbours.

Here are two simple examples.

Implementation

For 1-context-sensitive rules we have to slightly modify our evolution function.

```
SS2EvolveList[rule_, init_List, t_Integer] :=
 NestList[Flatten[Partition[#, 2, 1] /. rule] &, init, t]
```

Example 1

NKS, *p. 85 (a)*

```
SS2EvolveList[
 {
  {1, 1} → {0, 1},
  {1, 0} → {1, 0},
  {0, 1} → 0,
  {0, 0} → {0, 1}
 },
 {0, 1, 1, 0}, 8]
```

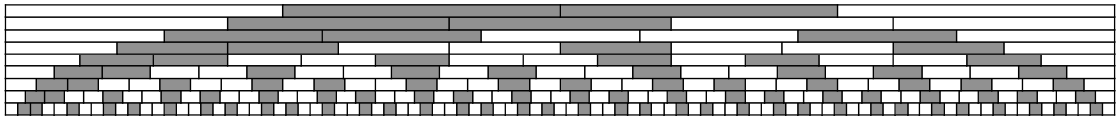
```
{{0, 1, 1, 0}, {0, 0, 1, 1, 0}, {0, 1, 0, 0, 1, 1, 0},
 {0, 1, 0, 0, 1, 0, 0, 1, 1, 0}, {0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0},
 {0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0},
 {0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1,
  0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0}, {0, 1, 0, 0, 1, 0, 1, 0, 0, 1,
  0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1,
  0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0},
 {0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1,
  0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0,
  1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0,
  1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0}}
```

```
% // ColumnForm
```

```
{0, 1, 1, 0}
{0, 0, 1, 1, 0}
{0, 1, 0, 0, 1, 1, 0}
{0, 1, 0, 0, 1, 0, 0, 1, 1, 0}
{0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0}
{0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0}
{0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0,
{0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0,
{0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0,
```

```
showRectangleEvolution[%%];
```

```
showRectangleEvolutionStrech[%%%];
```



Example 2

NKS, *p. 85 (b)*

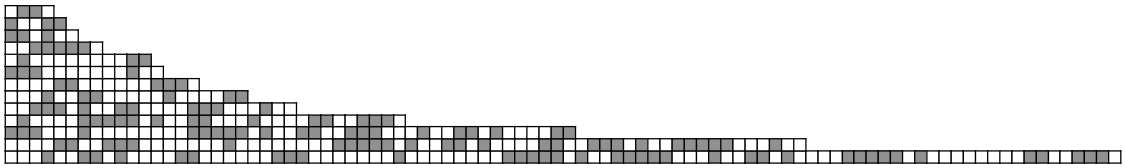
```
SS2EvolveList[
 {
  {1, 1} → {0, 0},
  {1, 0} → {1, 1},
  {0, 1} → 1,
  {0, 0} → 0
 },
 {0, 1, 1, 0}, 12]
```

```
{{0, 1, 1, 0}, {1, 0, 0, 1, 1}, {1, 1, 0, 1, 0, 0}, {0, 0, 1, 1, 1, 1, 1, 0},
 {0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1}, {1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
 {0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0},
 {0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1},
 {0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 0},
 {0, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0,
  0, 1, 1, 1, 0}, {1, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0,
  1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1},
 {0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0,
  0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 1,
  1, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 1, 0, 0},
 {0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1,
  1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1,
  0, 1, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0,
  1, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0}}
```

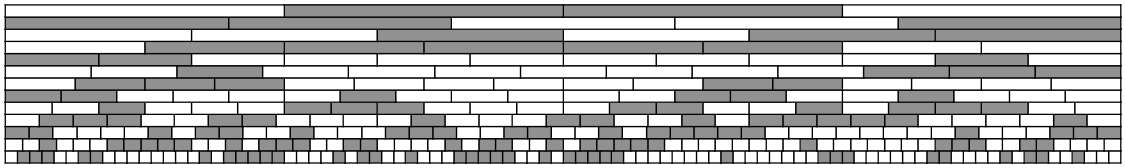
```
% // ColumnForm
```

```
{0, 1, 1, 0}
{1, 0, 0, 1, 1}
{1, 1, 0, 1, 0, 0}
{0, 0, 1, 1, 1, 1, 1, 0}
{0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1}
{1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0}
{0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0}
{0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1}
{0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 0}
{0, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0,
{1, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1,
{0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1,
{0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0,
```

```
showRectangleEvolution[%%];
```



```
showRectangleEvolutionStrech[%%%];
```



Example 3: Creation and Destruction

NKS, *p. 86*

The following rule system allows both for creation and destruction of elements.

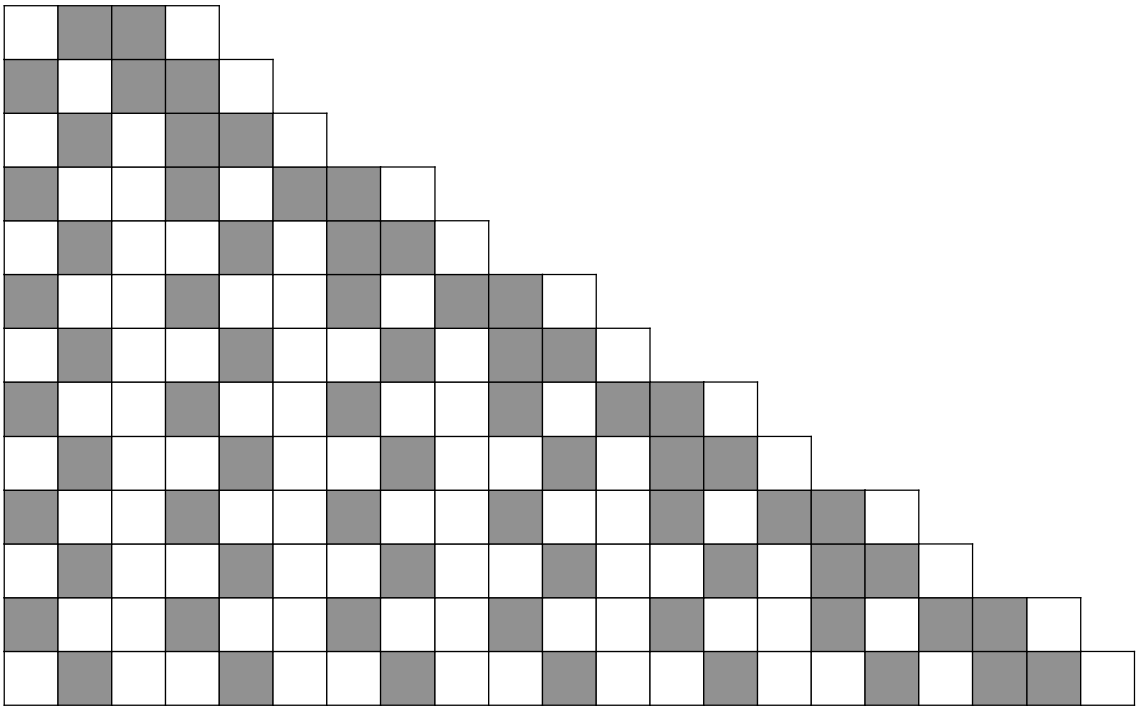
```
SS2EvolveList[
 {
  {1, 1} → {1, 1},
  {1, 0} → {0},
  {0, 1} → {1, 0},
  {0, 0} → {}
 },
 {0, 1, 1, 0}, 12]
```

```
{{0, 1, 1, 0}, {1, 0, 1, 1, 0}, {0, 1, 0, 1, 1, 0}, {1, 0, 0, 1, 0, 1, 1, 0},
 {0, 1, 0, 0, 1, 0, 1, 1, 0}, {1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0},
 {0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0}, {1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0},
 {0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0},
 {1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0},
 {0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0},
 {1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0},
 {0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0}}
```

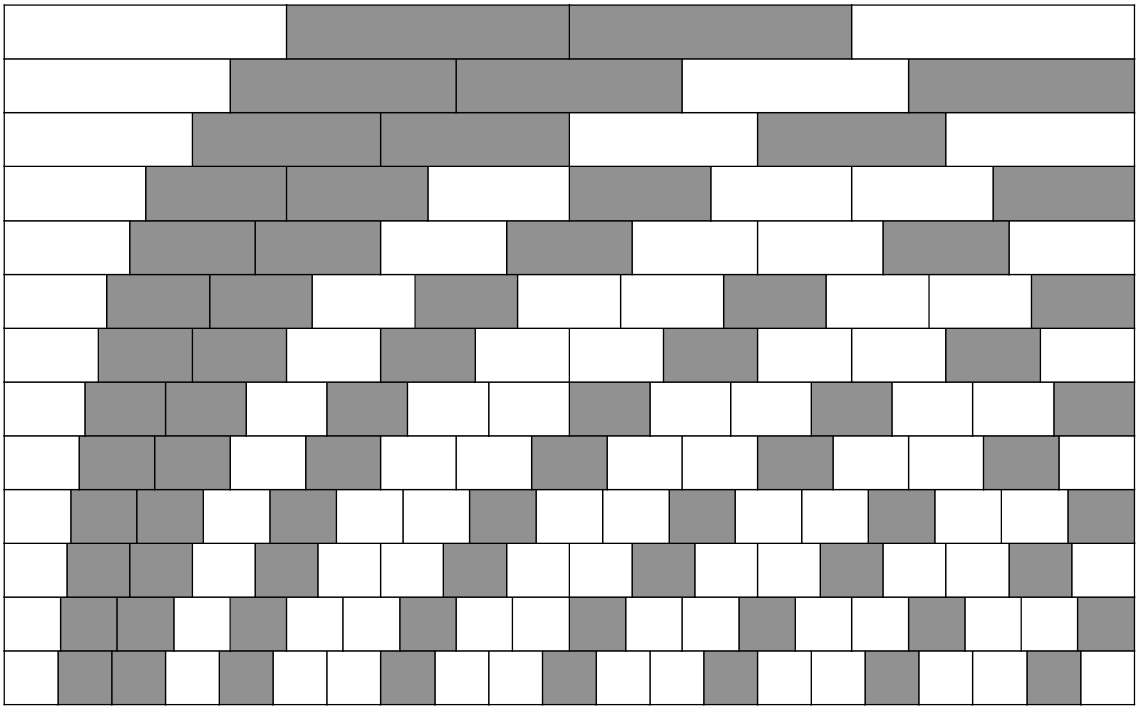
```
% // ColumnForm
```

```
{0, 1, 1, 0}
{1, 0, 1, 1, 0}
{0, 1, 0, 1, 1, 0}
{1, 0, 0, 1, 0, 1, 1, 0}
{0, 1, 0, 0, 1, 0, 1, 1, 0}
{1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0}
{0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0}
{1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0}
{0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0}
{1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0}
{0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0}
{1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0}
{0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0}
```

```
showRectangleEvolution[%%];
```



```
showRectangleEvolutionStrech[%%%];
```



Example 4: Creation and Destruction (3 and 4 possible colours)

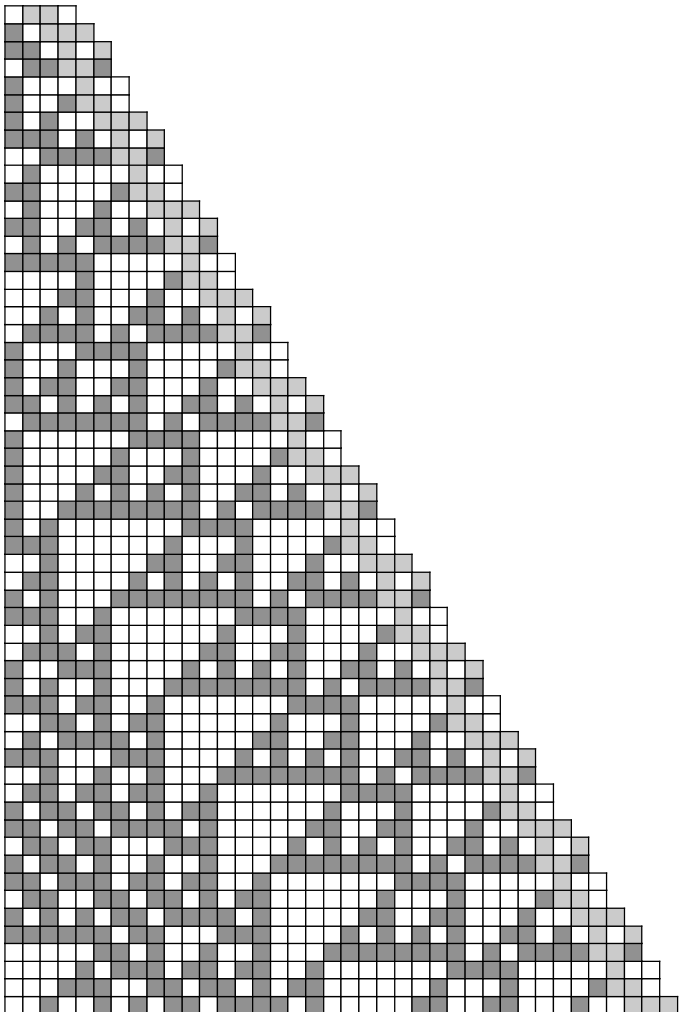
NKS, *p. 87*

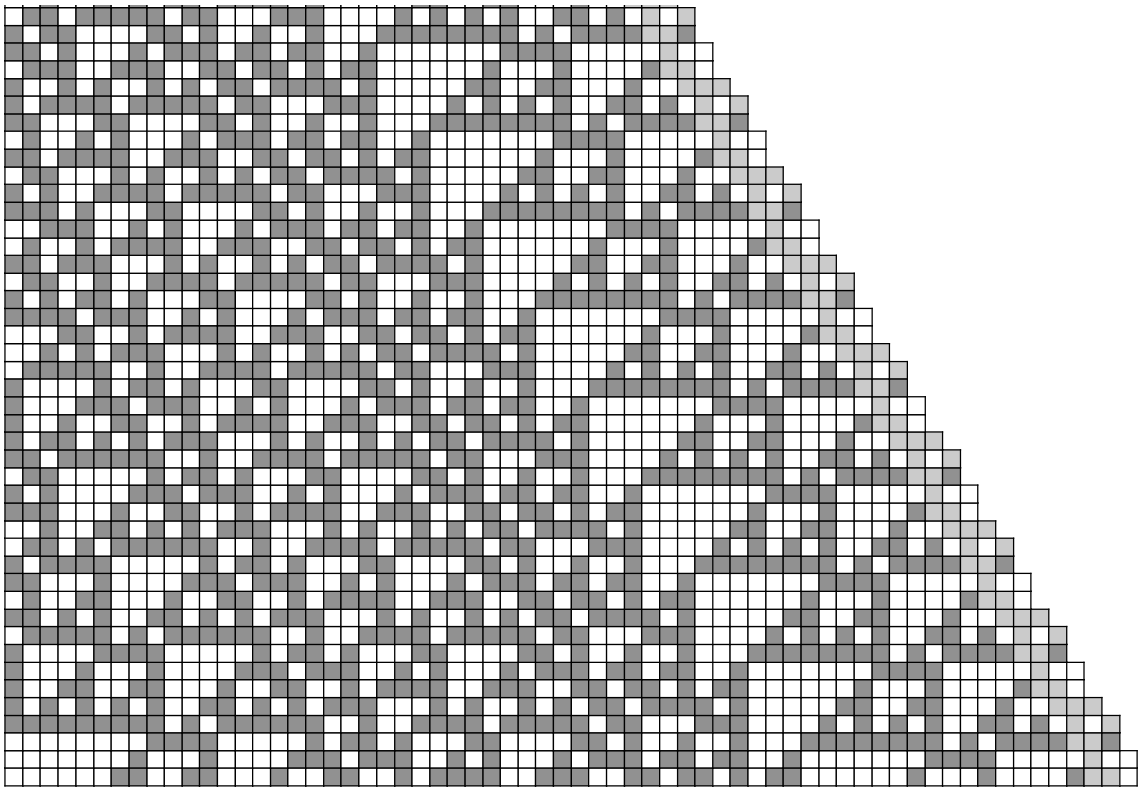
Here are some examples of substitution systems that have three or four possible colours for each element. The rules shown lead to slow growth in the total number of elements.

■ Growth (a)

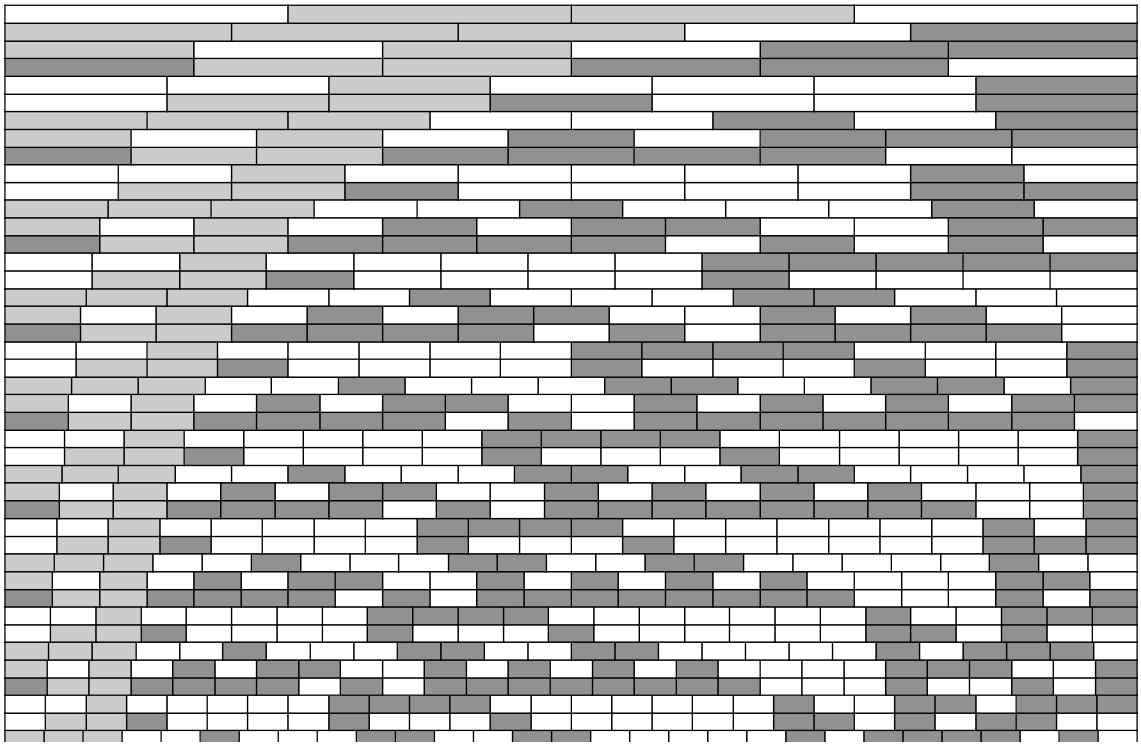
```
ssEvolve = SS2EvolveList[
   {
    {1, 1} → {0},
    {1, .5} → {0},
    {1, 0} → {1},
    {.5, 1} → {0, 0},
    {.5, .5} → {0, .5},
    {.5, 0} → {.5, .5},
    {0, 1} → {1},
    {0, .5} → {1},
    {0, 0} → {0}
   },
   {0, .5, .5, 0}, 100];
```

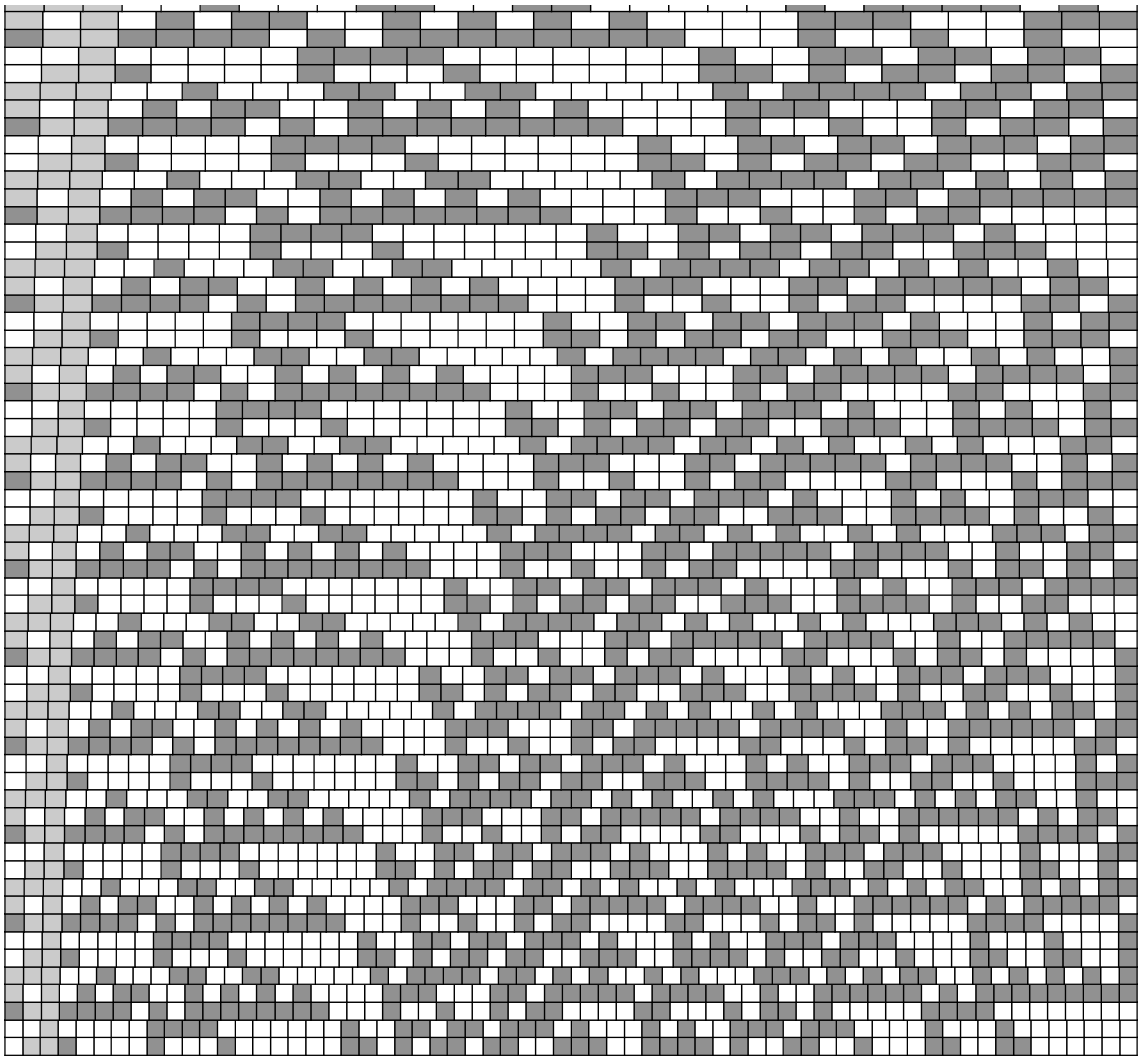
```
showRectangleEvolution[ssEvolve];
```





```
showRectangleEvolutionStrech[ssEvolve];
```

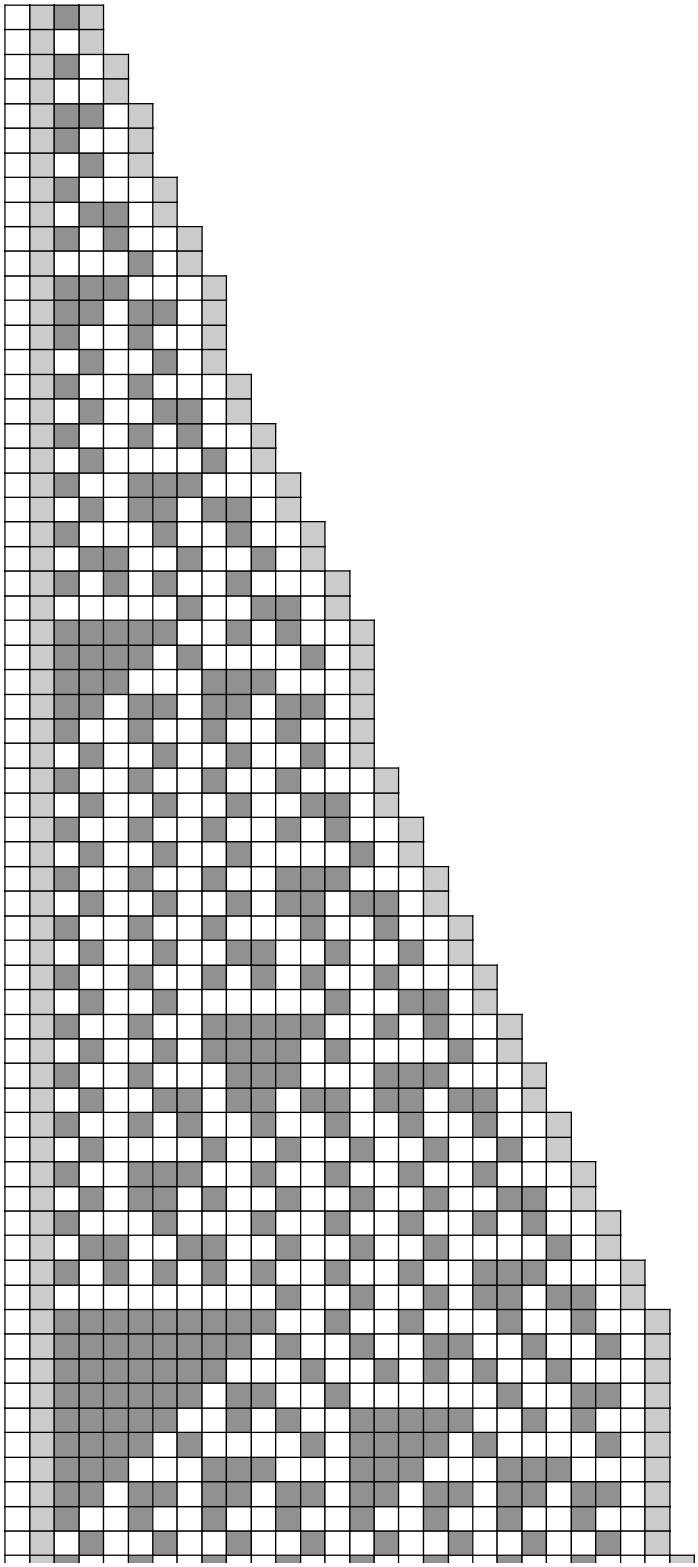


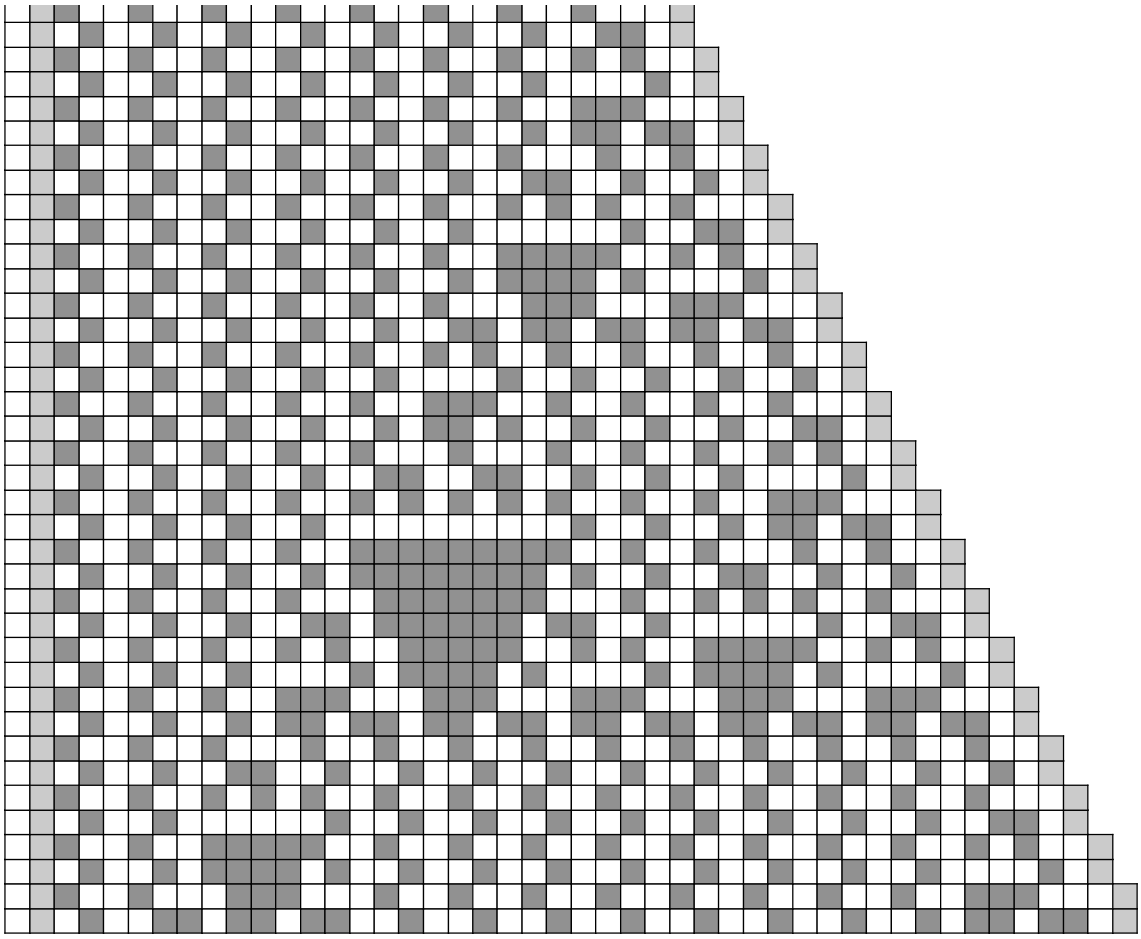


■ Growth (b)

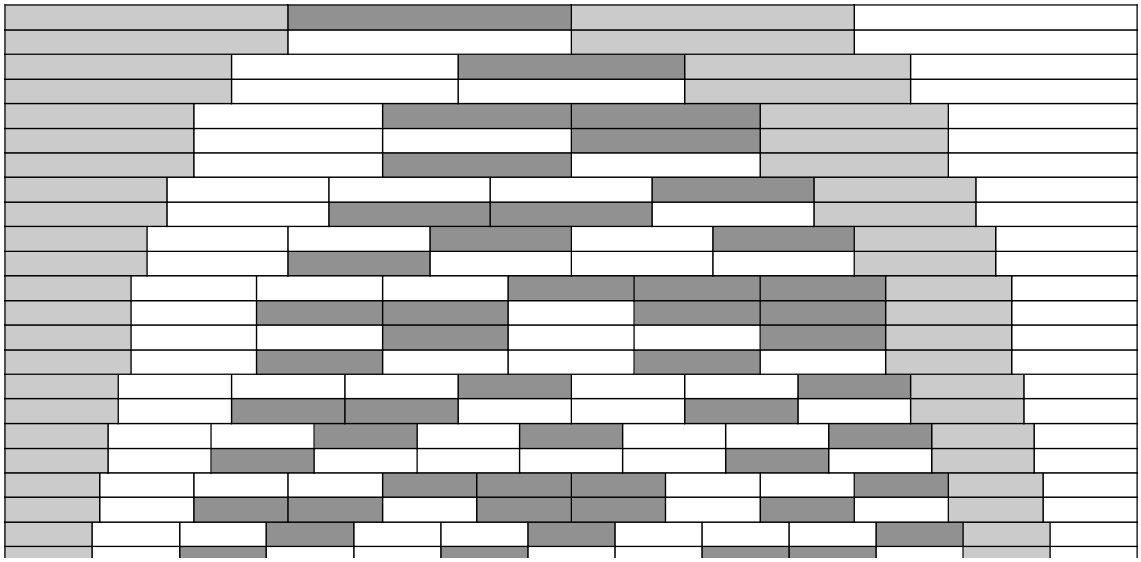
```
ssEvolve = SS2EvolveList[
   {
    {1, 1} → {1},
    {1, .5} → {0, .5},
    {1, 0} → {0},
    {.5, 1} → {},
    {.5, .5} → {},
    {.5, 0} → {1},
    {0, 1} → {0},
    {0, .5} → {0, .5},
    {0, 0} → {1}
   },
   {0, .5, 1, .5}, 100];
```

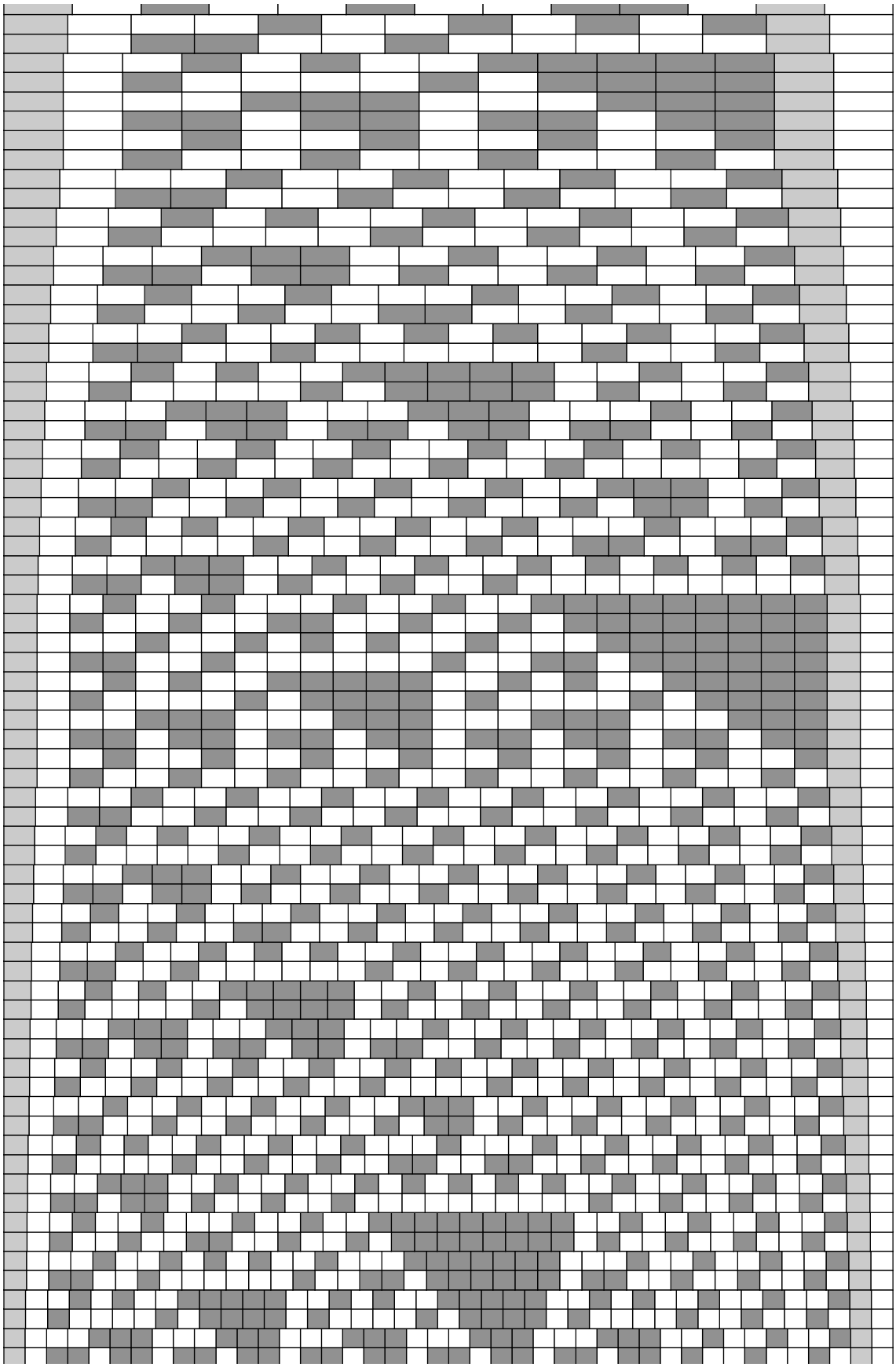
```
showRectangleEvolution[ssEvolve];
```

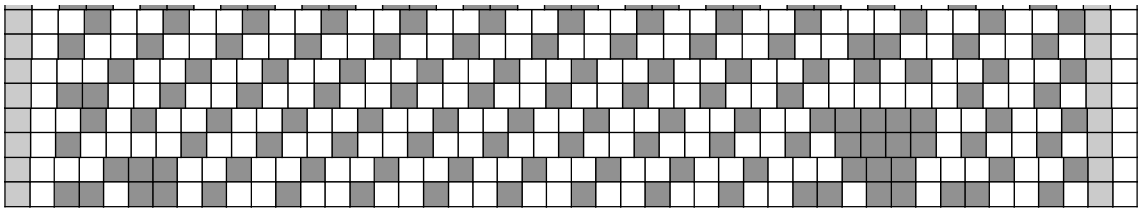




```
showRectangleEvolutionStrech[ssEvolve];
```



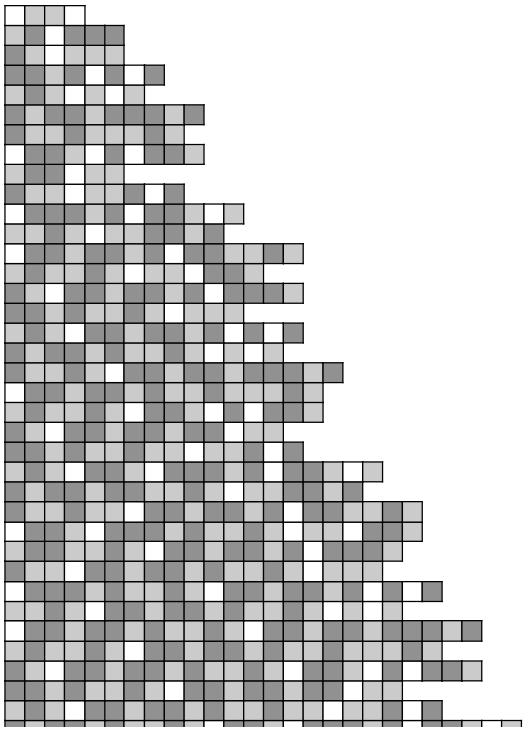


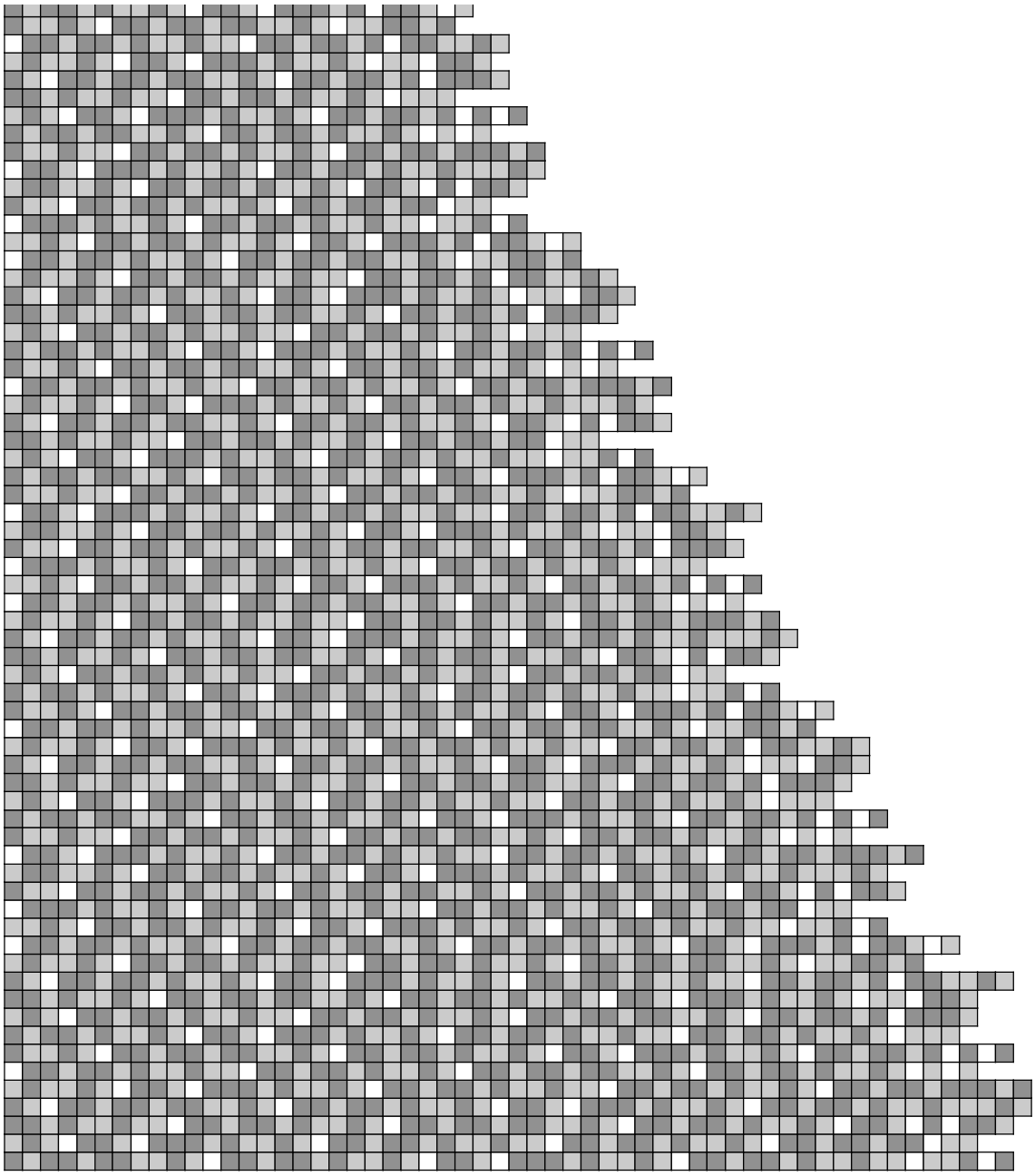


■ Growth (c)

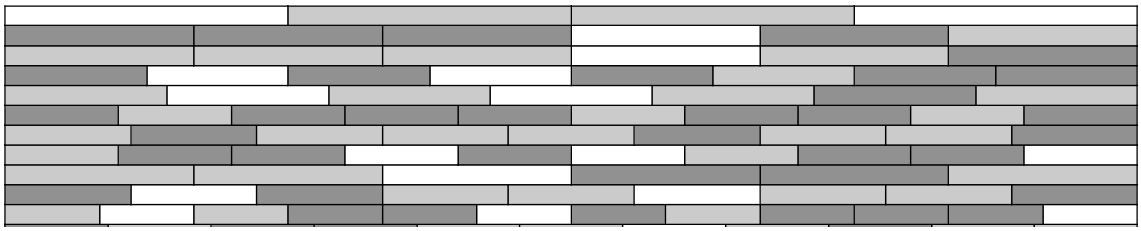
```
ssEvolve = SS2EvolveList[
   {
    {1, 1} → {.5},
    {1, .5} → {},
    {1, 0} → {0, .5},
    {.5, 1} → {1, .5},
    {.5, .5} → {0, 1},
    {.5, 0} → {1, 1},
    {0, 1} → {},
    {0, .5} → {.5, 1},
    {0, 0} → {0}
   },
   {0, .5, .5, 0}, 100];
```

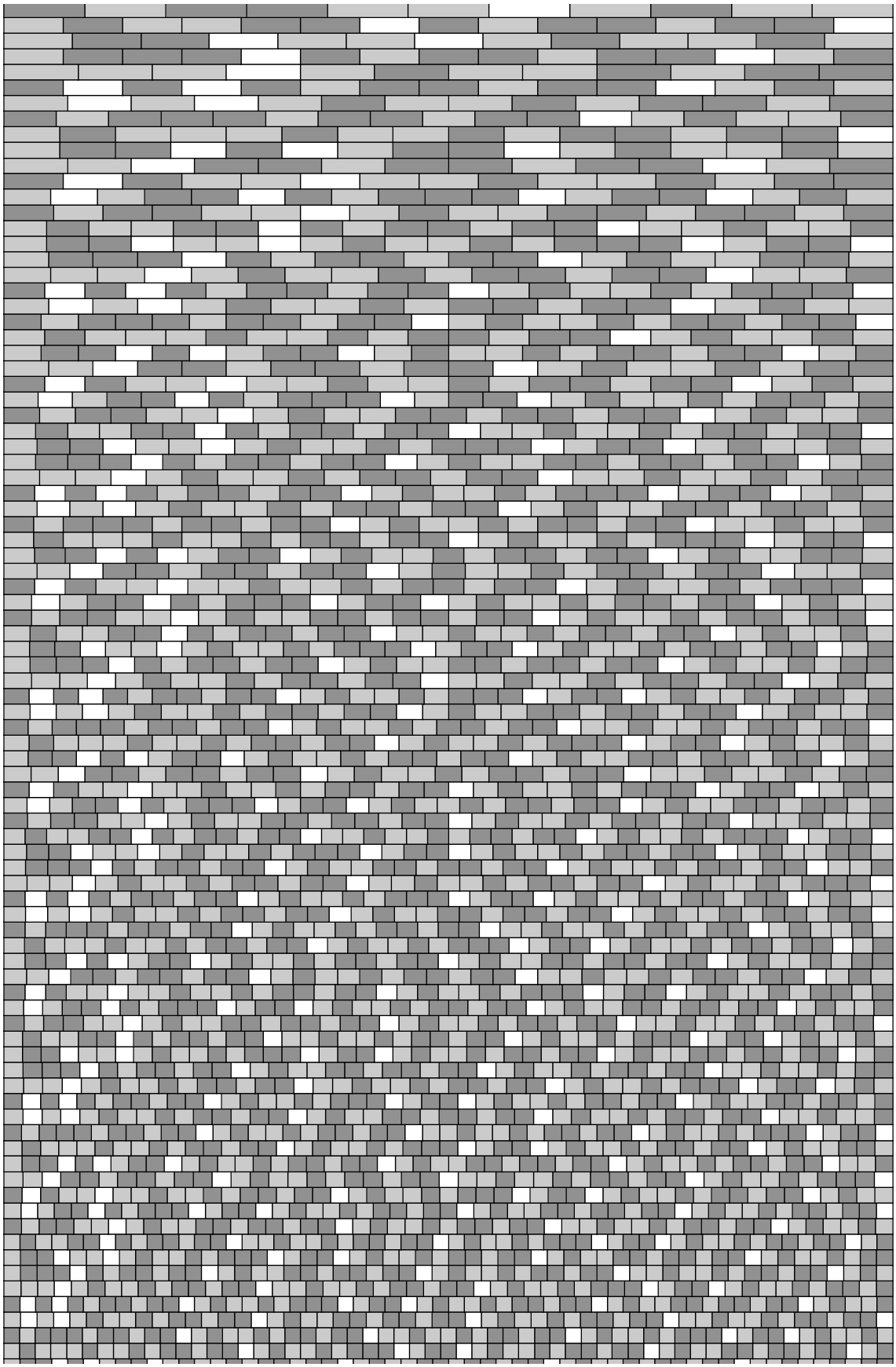
```
showRectangleEvolution[ssEvolve];
```





```
showRectangleEvolutionStrech[ssEvolve];
```

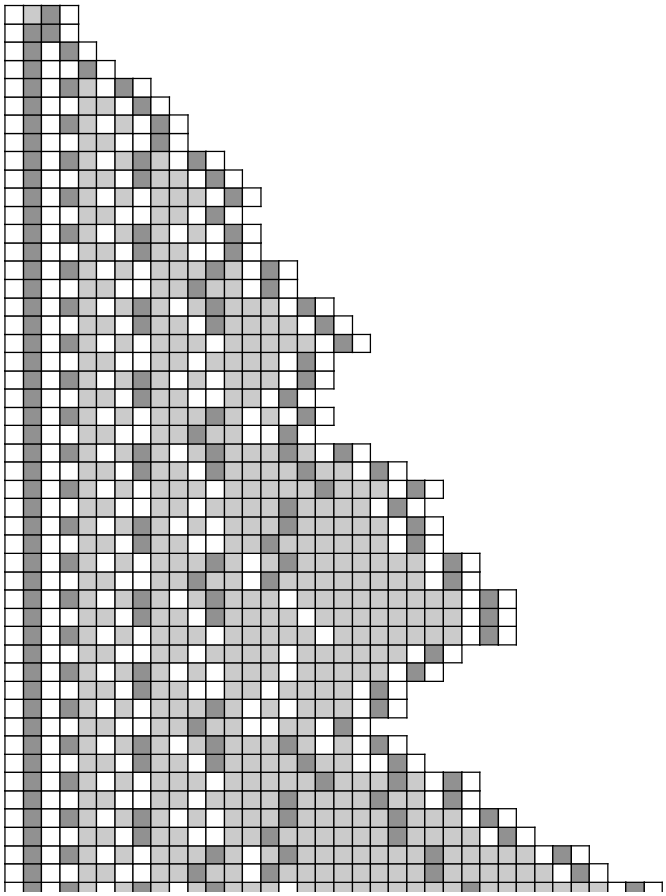


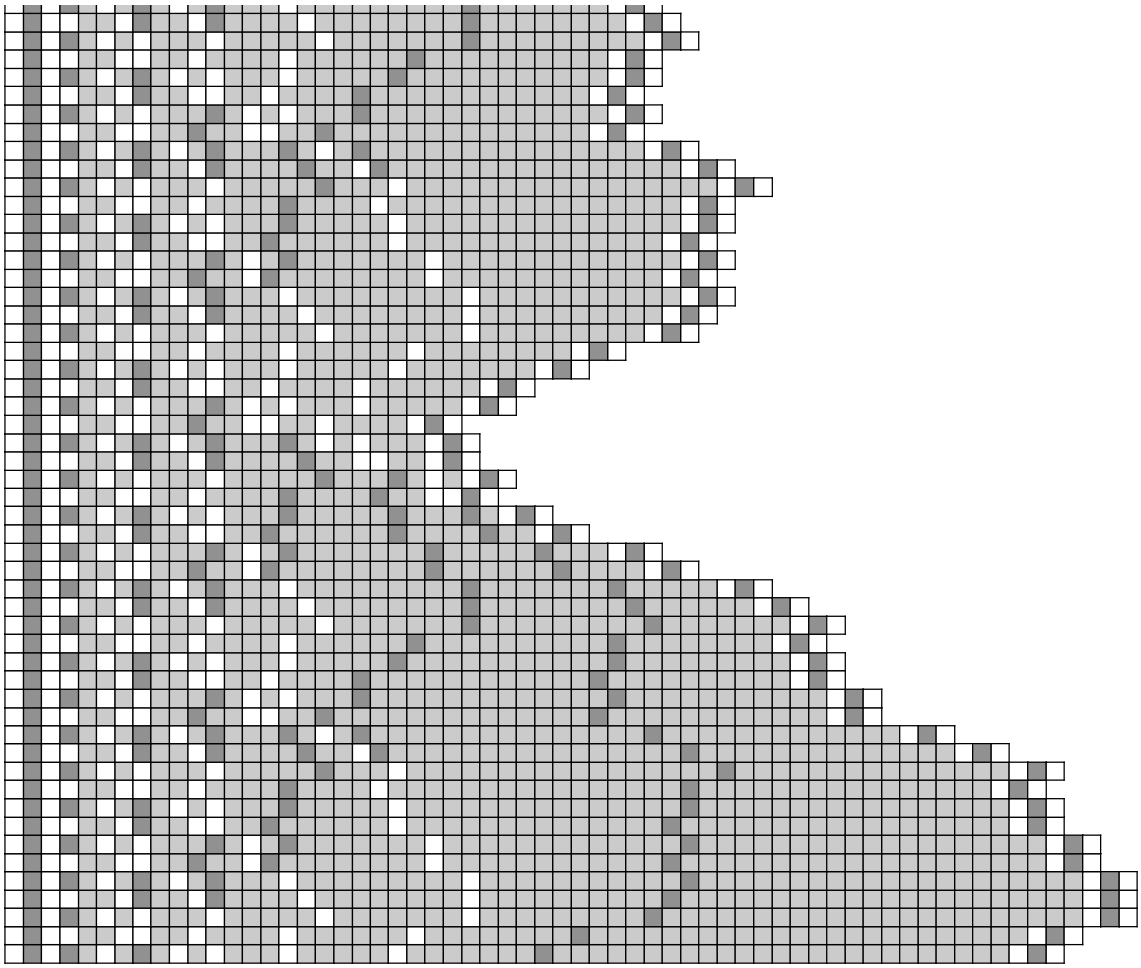


■ Growth (d)

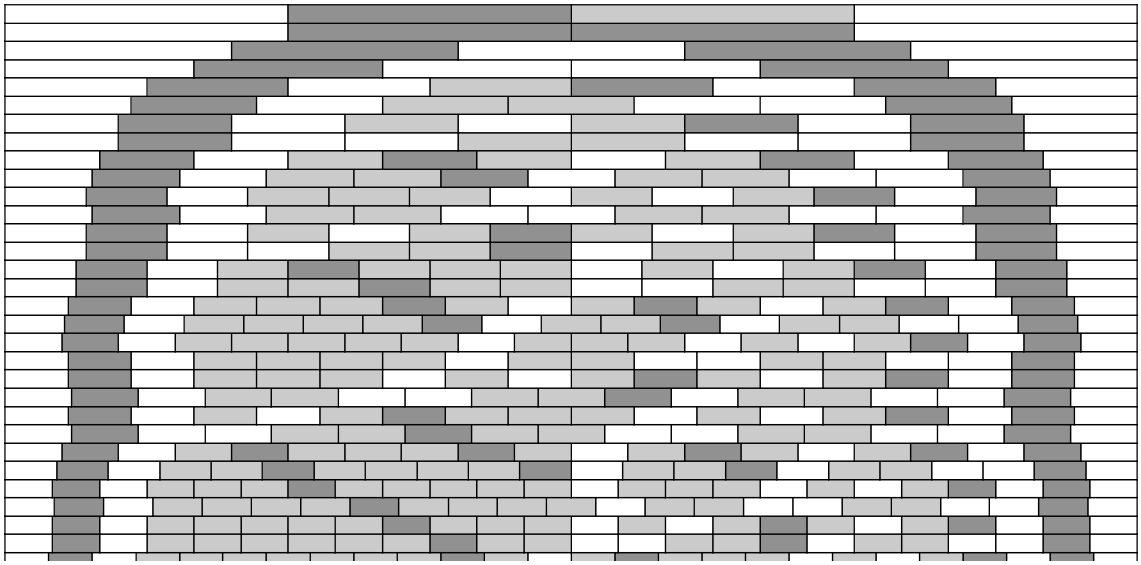
```
ssEvolve = SS2EvolveList[
   {
    {1, 1} → {1, 0},
    {1, .5} → {.5, .5},
    {1, 0} → {1, 0},
    {.5, 1} → {1},
    {.5, .5} → {.5},
    {.5, 0} → {},
    {0, 1} → {0},
    {0, .5} → {0},
    {0, 0} → {1, .5}
   },
   {0, .5, 1, 0}, 100];
```

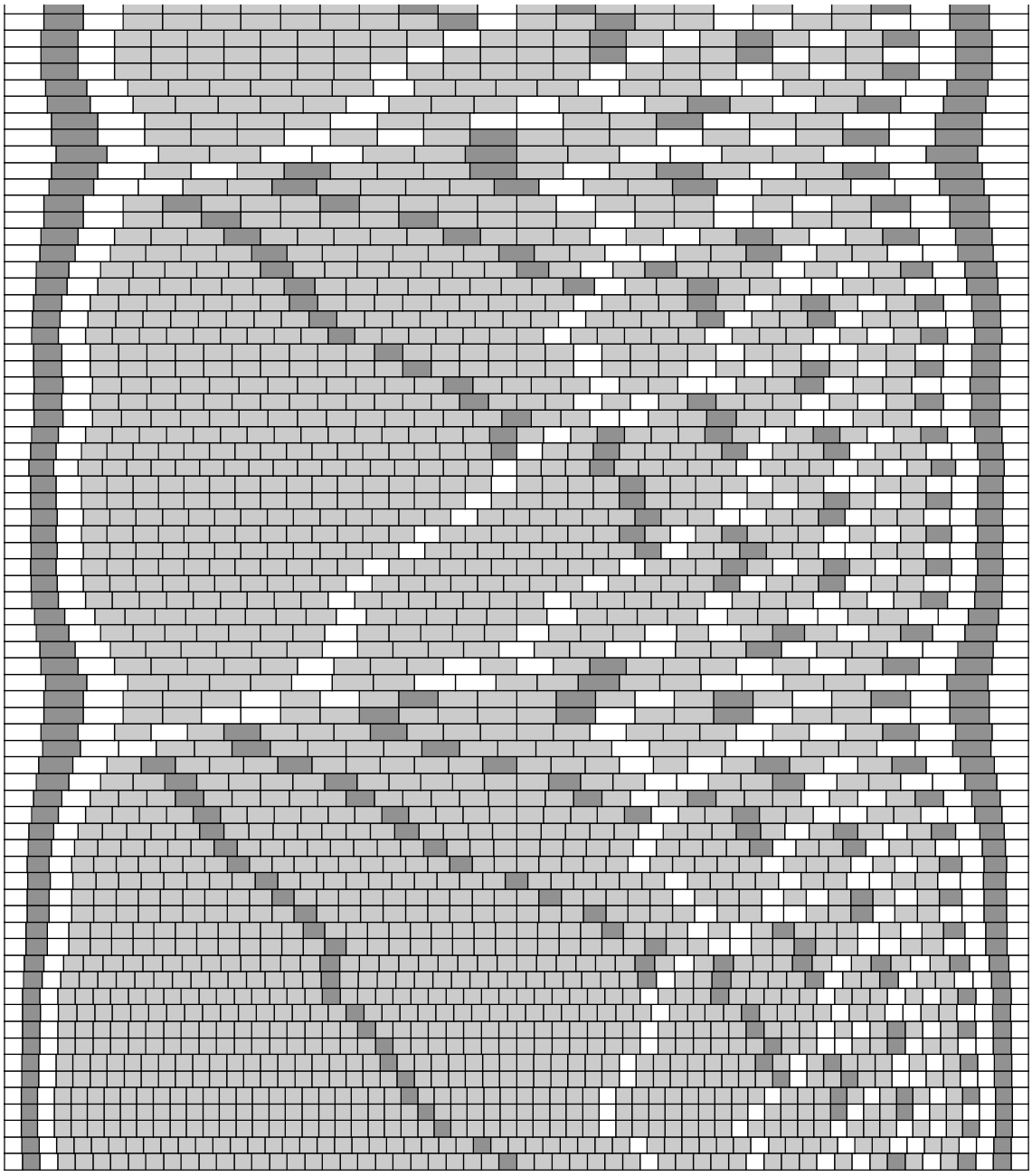
```
showRectangleEvolution[ssEvolve];
```





```
showRectangleEvolutionStrech[ssEvolve];
```



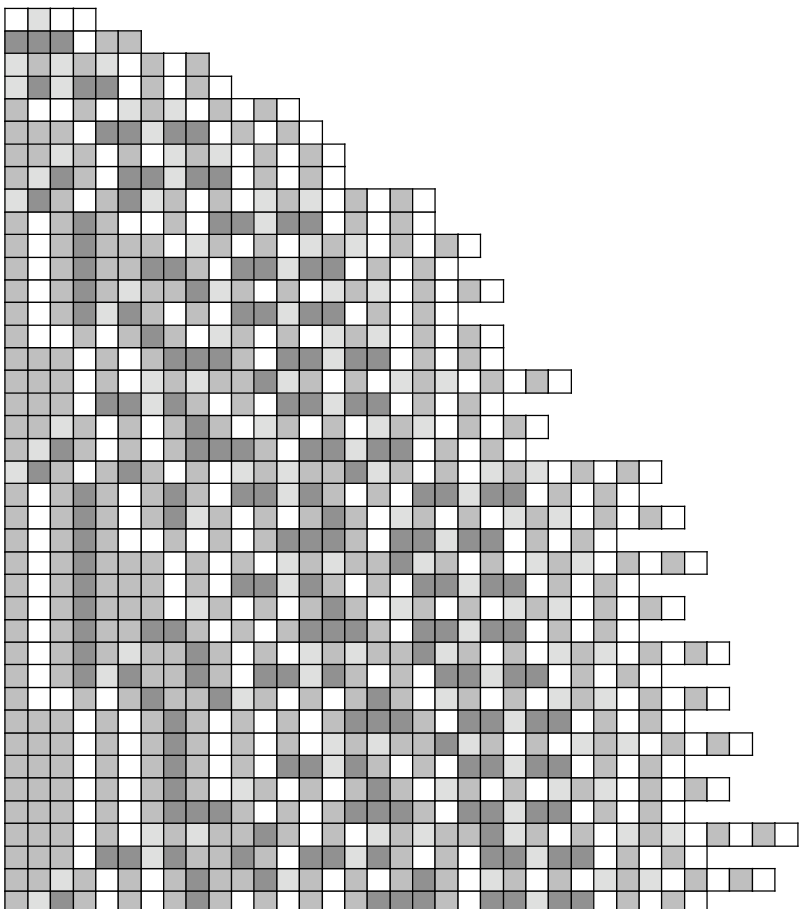


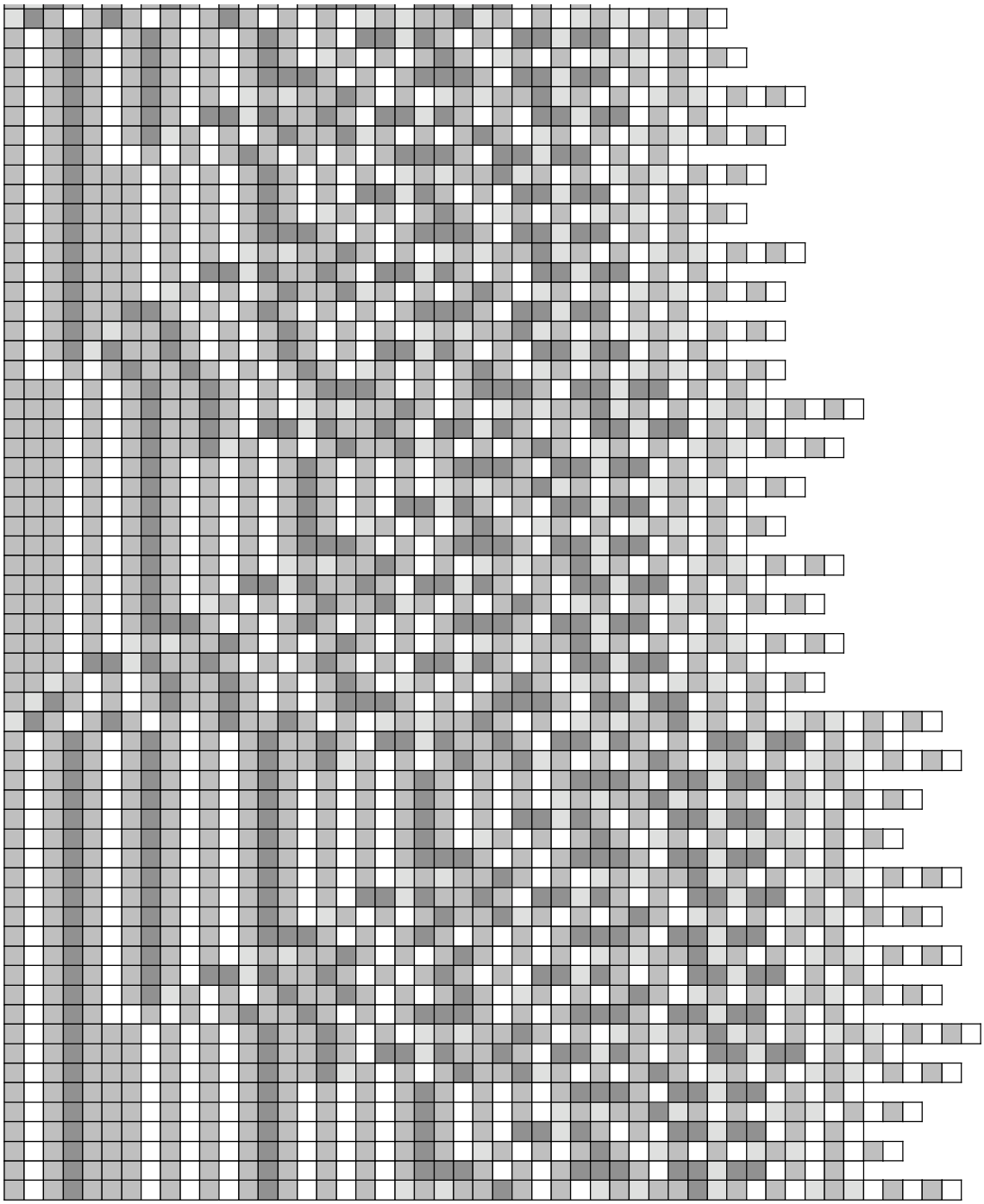
■ Growth (e)

4 colours

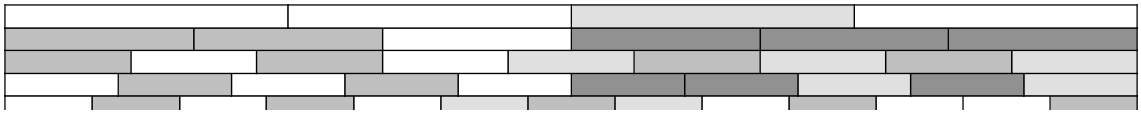
```
ssEvolve = SS2EvolveList[
   {
    {1, 1} → {.3, .6},
    {1, .6} → {.6, 1},
    {1, .3} → {0},
    {1, 0} → {.3, 0},
    {.6, 1} → {},
    {.6, .6} → {.6},
    {.6, .3} → {.3, 1},
    {.6, 0} → {},
    {.3, 1} → {.6, 0},
    {.3, .6} → {},
    {.3, .3} → {.3},
    {.3, 0} → {1, 0},
    {0, 1} → {},
    {0, .6} → {.6, 0},
    {0, .3} → {1, 1},
    {0, 0} → {.6, .6}
   },
   {0, .3, 0, 0}, 100];
```

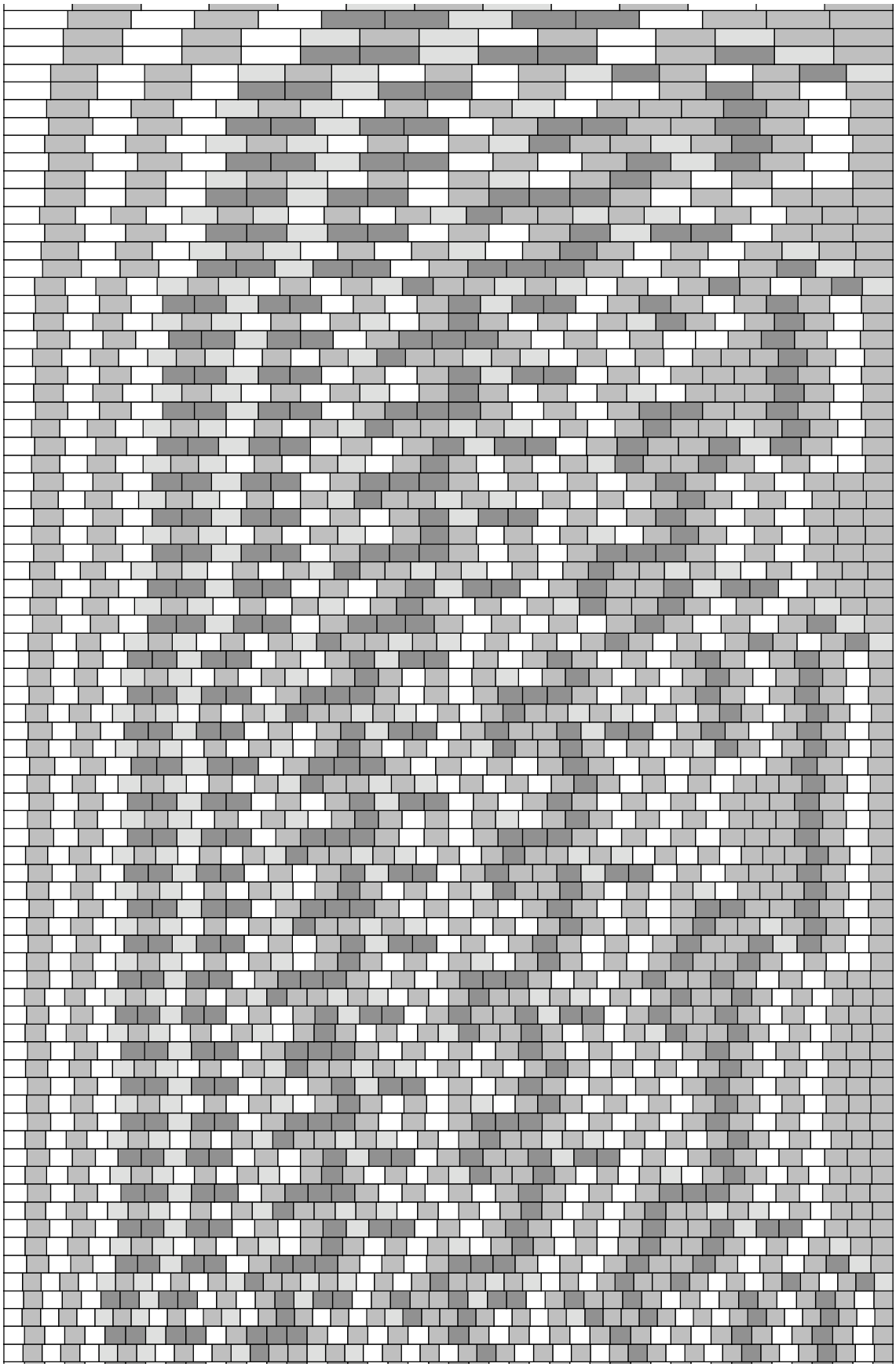
```
showRectangleEvolution[ssEvolve];
```

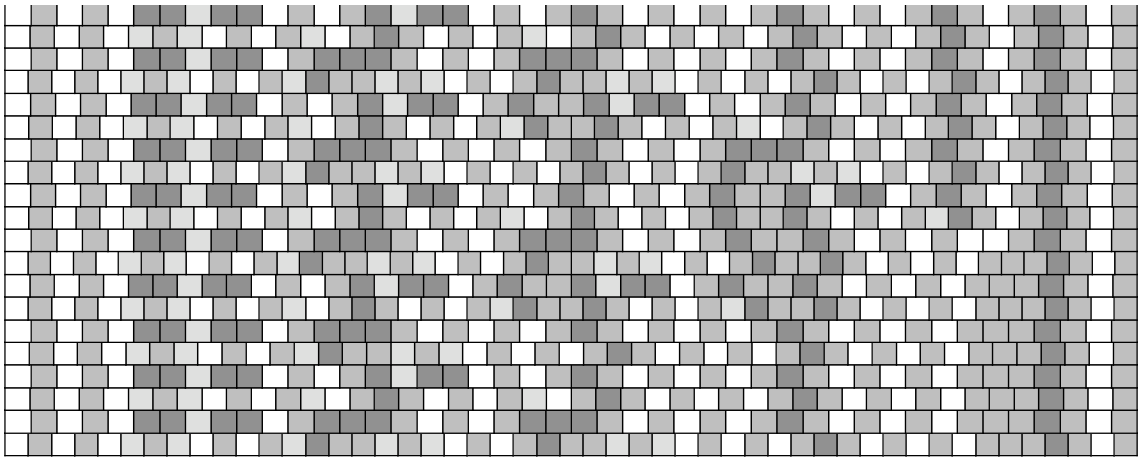




```
showRectangleEvolutionStrech[ssEvolve];
```





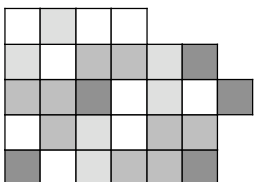


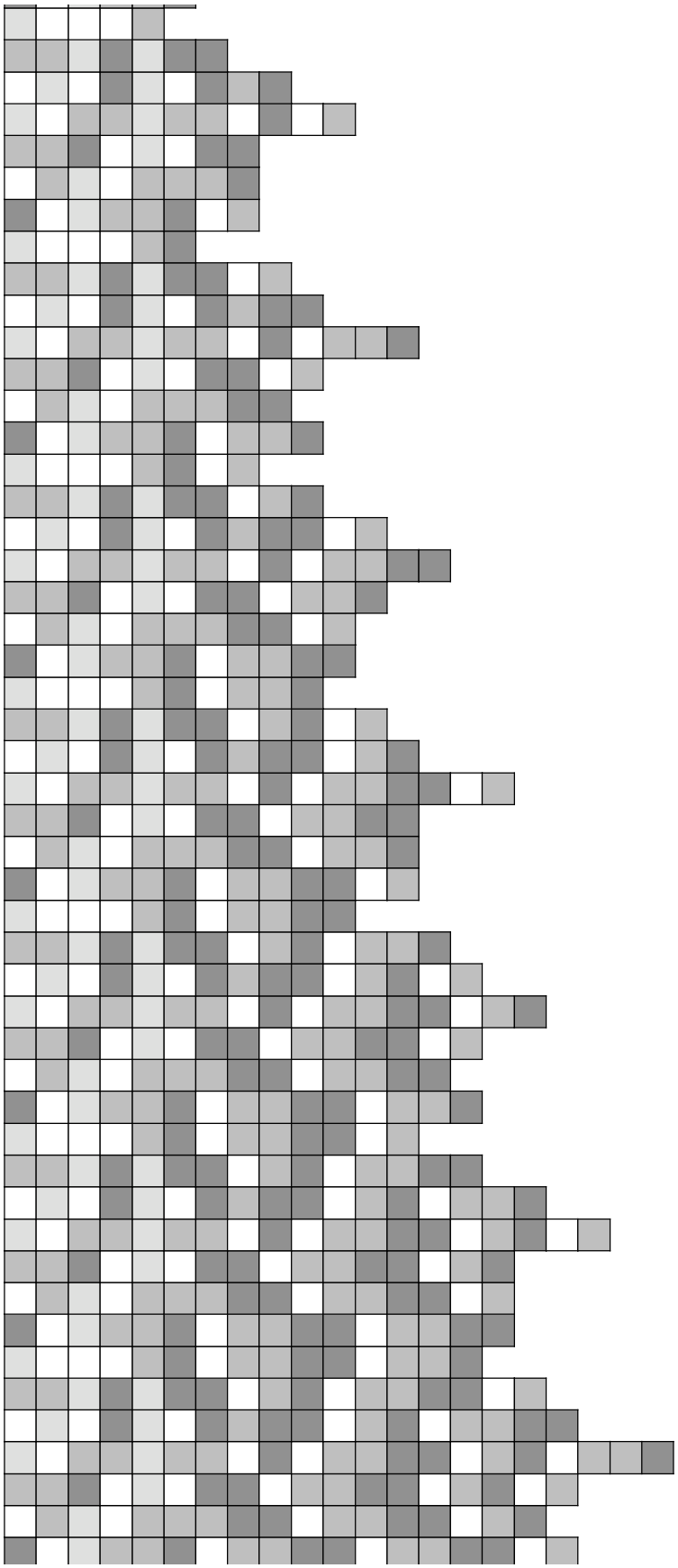
■ Growth (f)

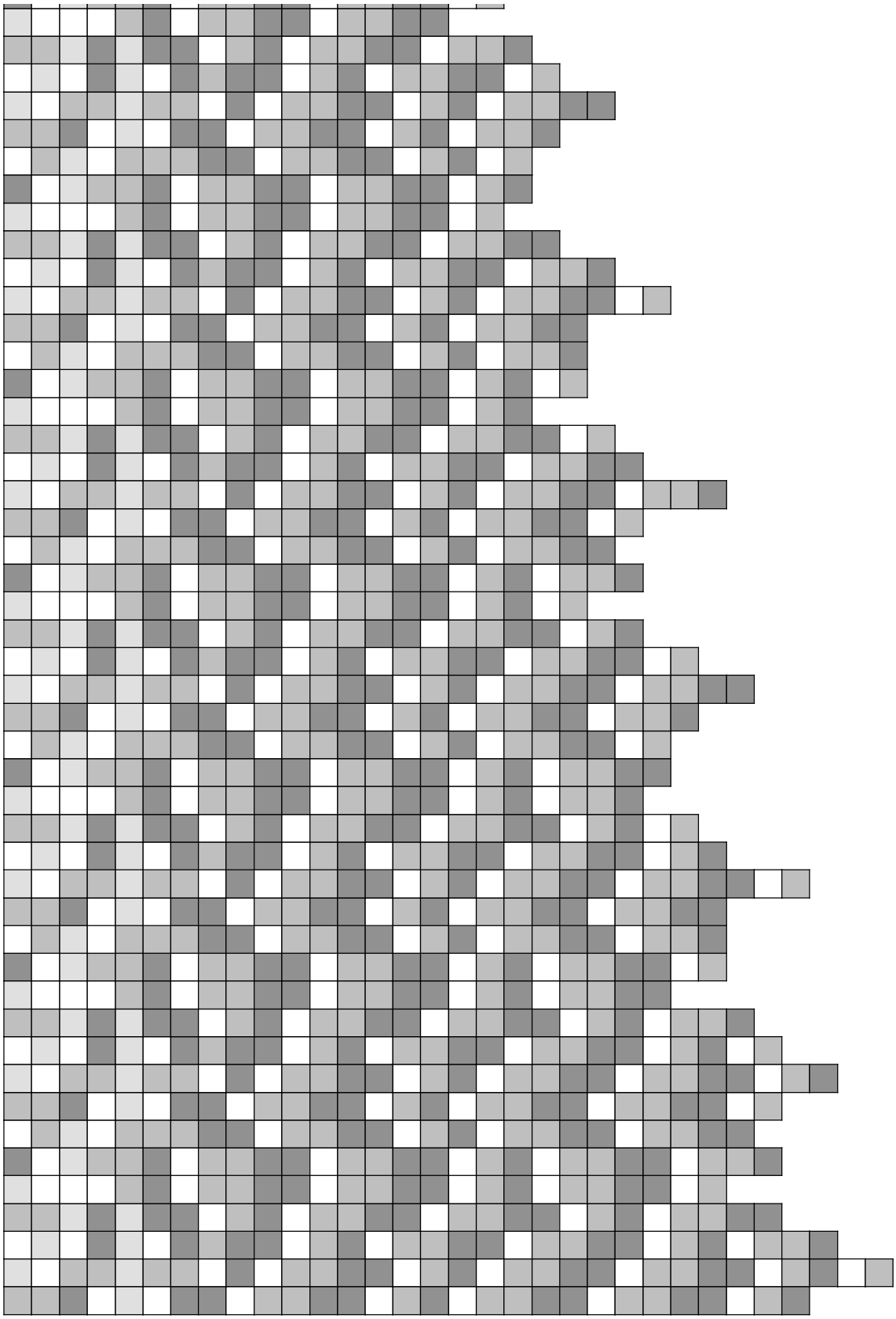
4 colours

```
ssEvolve = SS2EvolveList[
   {
    {1, 1} → {.6, 1},
    {1, .6} → {0, 1},
    {1, .3} → {.3},
    {1, 0} → {},
    {.6, 1} → {0, .6},
    {.6, .6} → {},
    {.6, .3} → {0, .3},
    {.6, 0} → {1},
    {.3, 1} → {0, 1},
    {.3, .6} → {0},
    {.3, .3} → {.6, .3},
    {.3, 0} → {.6, .6},
    {0, 1} → {},
    {0, .6} → {1},
    {0, .3} → {.3, 0},
    {0, 0} → {.3, 1}
   },
   {0, .3, 0, 0}, 100];
```

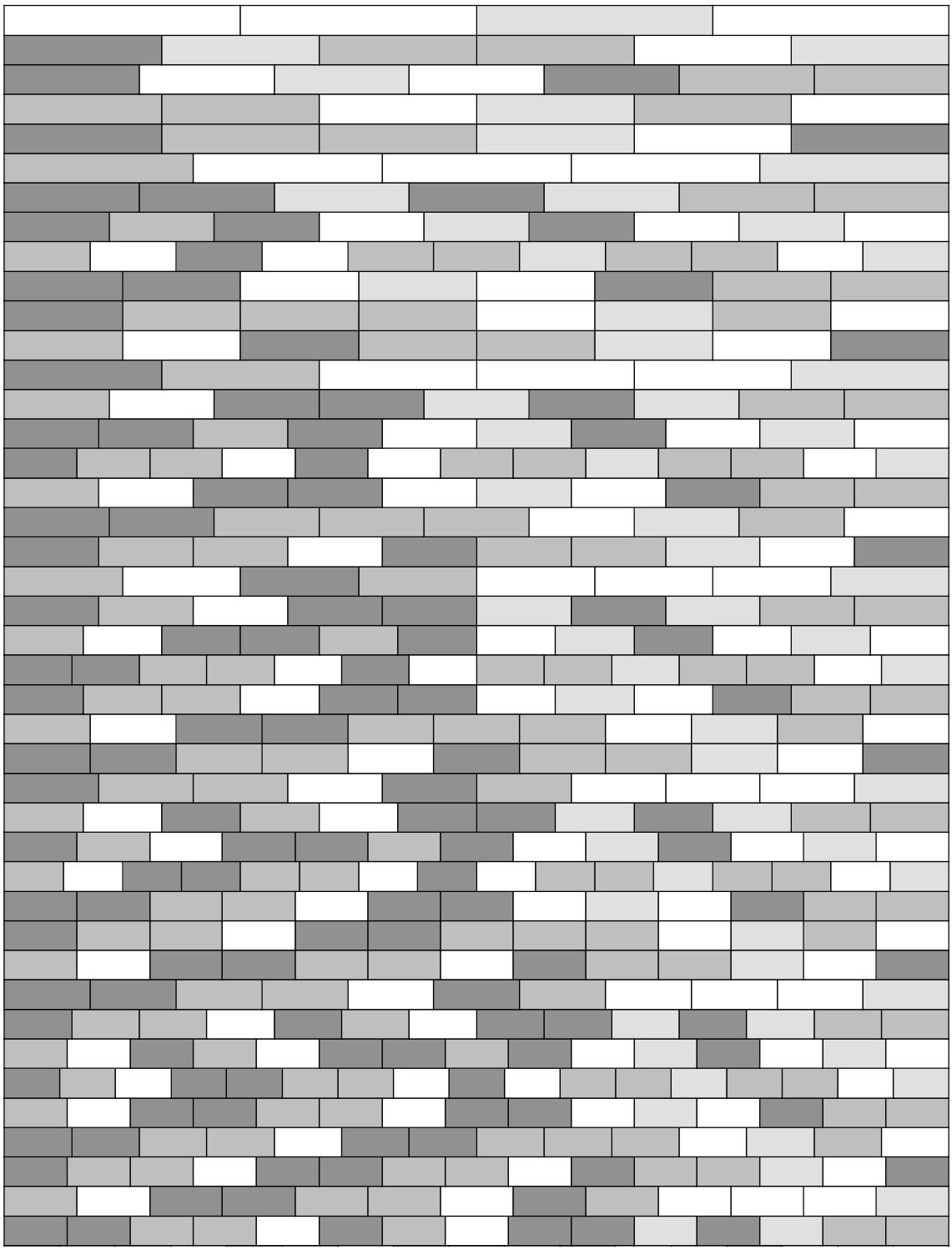
```
showRectangleEvolution[ssEvolve];
```

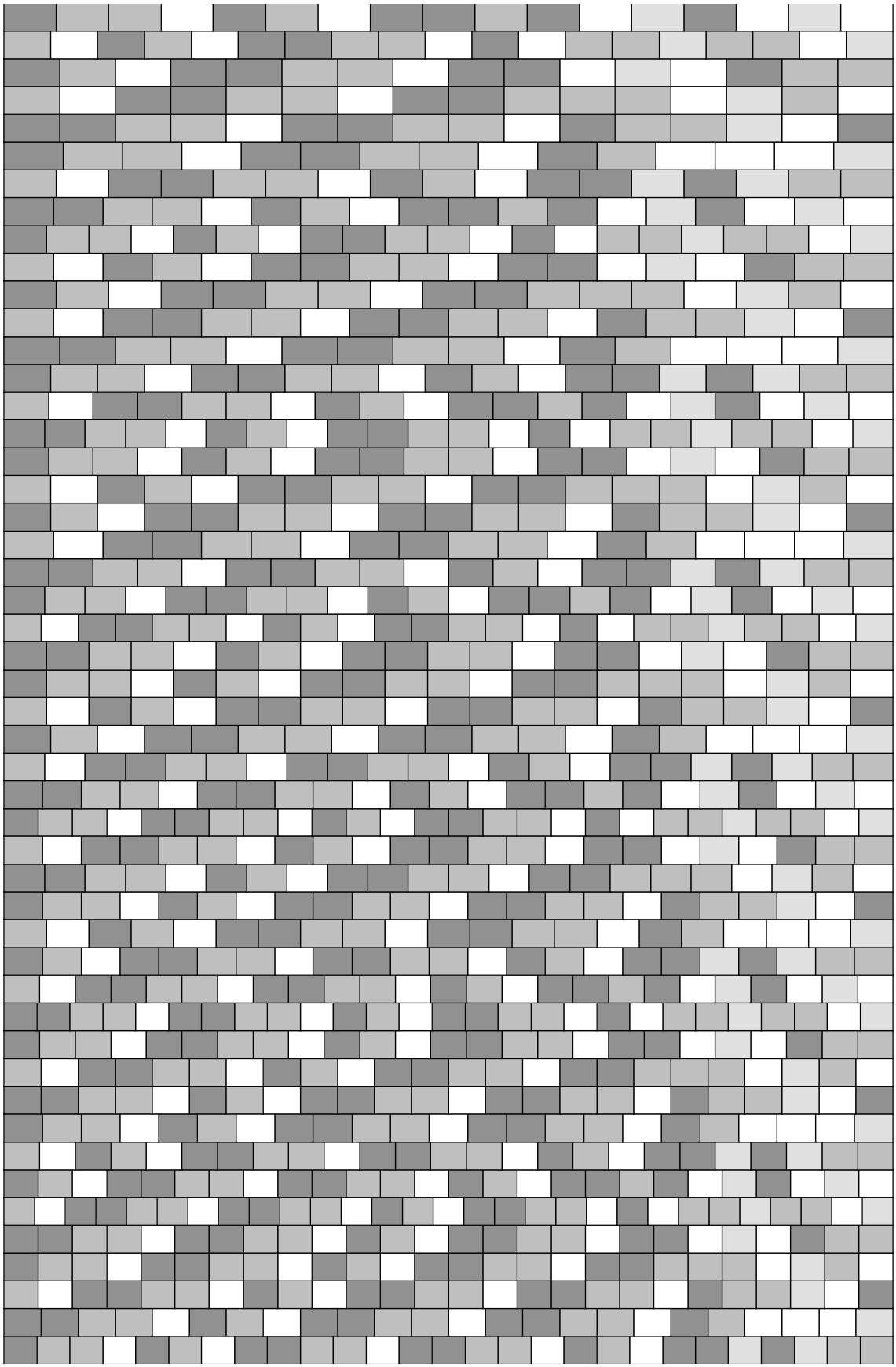


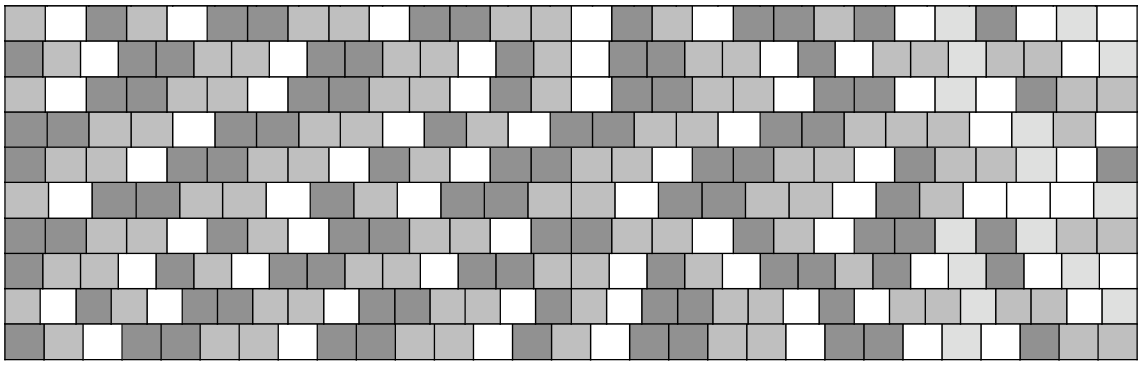




```
showRectangleEvolutionStrech[ssEvolve];
```







Code

Reference

S. Wolfram, A New Kind of Science, Wolfram Media, 2002.