

Point counting without points

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The rules of point counting

Philosophy of Drinfeld modules

Representation of Drinfeld modules

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What is point counting?

Naive approach

Counting solutions to an equation.

For algebraic varieties on a finite field: hard algorithmic problem.

Consider algebraic objects: *elliptic curves, abelian varieties*.

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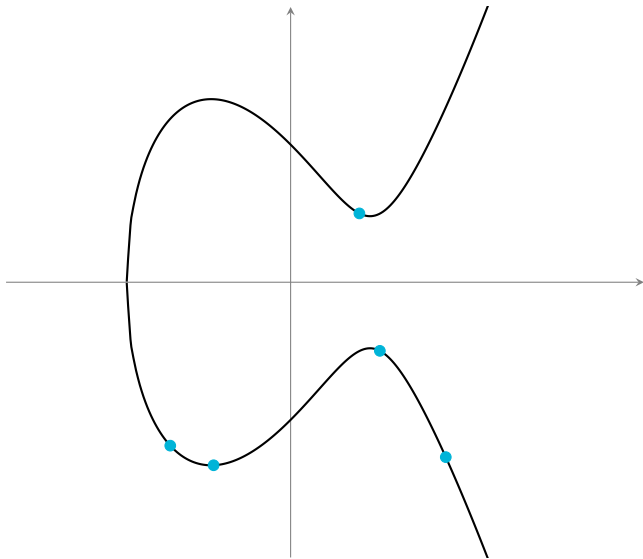
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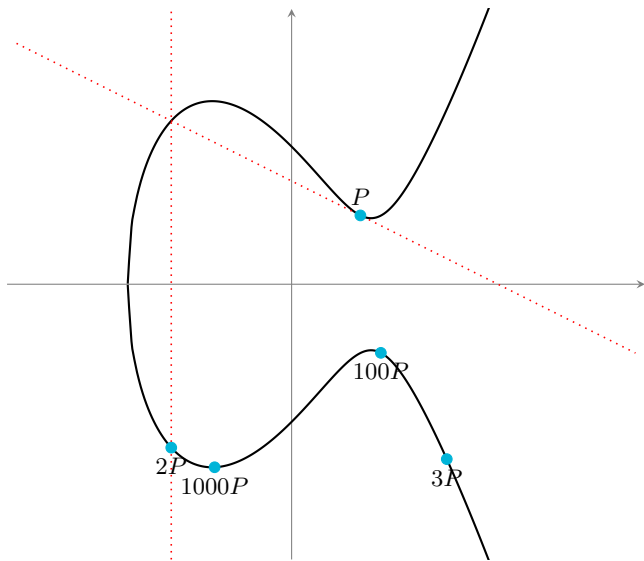
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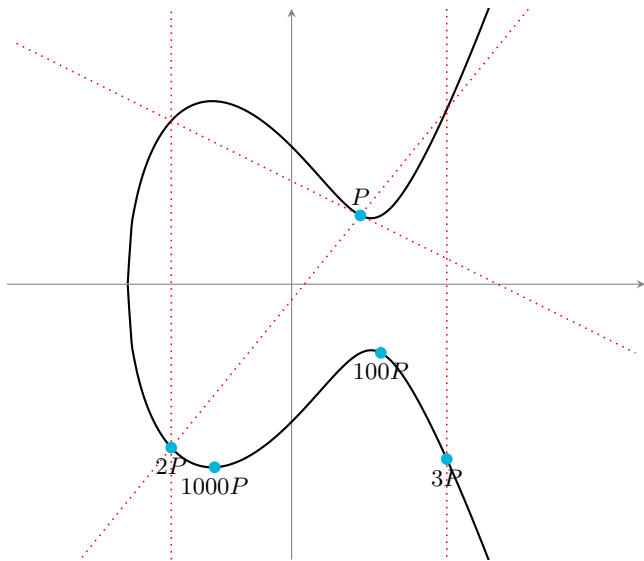
Algebraic structure



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Changing the rules

Let E be an elliptic curve. As an abelian group,

$$E(\mathbb{F}_q) \simeq \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_n\mathbb{Z}.$$

So

$$\#E(\mathbb{F}_q) = |d_1 \cdots d_n|.$$

Let R be a PID, M be a finite R -module. There are $m_1, \dots, m_\ell \in R$ s.t.:

$$M \simeq R/m_1R \times \cdots \times R/m_\ell R.$$

R-cardinality

Define the *R*-cardinality of M as

$$m_1 \cdots m_\ell.$$

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An alternative to $\mathbb{Z}$

Consider replacing $\mathbb{Z}$ by $R = \mathbb{F}_q[T]$!

Both Euclidean rings:

- $\mathbb{Z}$: number fields;
- $\mathbb{F}_q[T]$: function fields.

Advantages of function fields

- Unconditional results (*e.g.* GRH).
- Faster algorithms (*e.g.* factorization).
- Geometrical properties of function fields.
- And others: $\mathbb{F}_q$ -linearity, non-Archimedean analysis, etc.

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What are elliptic curves for $R = \mathbb{F}_q[T]$?

Module structure	
$\mathbb{Z}$ -module	$\mathbb{F}_q[T]$ -module
Torsion	
$(\mathbb{Z}/n\mathbb{Z})^2, p \nmid n$	$(\mathbb{F}_q[T]/a\mathbb{F}_q[T])^2, \mathfrak{p} \nmid a$
Endomorphism ring	
$\mathbb{Z}$, order in $\mathbb{Q}(\sqrt{-d})$, order in $\mathcal{B}_{p,\infty}$	Same, over the function field $\mathbb{F}_q(T)$

What objects play give this? Drinfeld modules!

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Analogies

$\mathbb{Z}$	$\mathbb{F}_q[T]$
$\mathbb{Q}$	$\mathbb{F}_q(T)$
Number fields (finite ext.)	Function fields (finite ext.)
$\mathbb{R}$	$\mathbb{R}_\infty = \mathbb{F}_q((\frac{1}{T}))$
$\mathbb{C}$	$\mathbb{C}_\infty = \text{completion of } \overline{\mathbb{R}_\infty}$
Elliptic curves	Drinfeld modules

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Our integers are polynomials.

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Applications of Drinfeld modules

Introduced by Drinfeld (*elliptic modules*) in 1977. First works by Carlitz.

Function field arithmetics

- Explicit class field theory and theory of complex multiplication.
- Geometric Langlands program.
- Others: exponential and logarithm functions, Drinfeld modular forms, etc.

Computer algebra

State-of-the art factorization in $\mathbb{F}_q[T]$, by computing Hasse invariants (Doliskani-Narayanan-Schost, 2021).

Cryptography

Drinfeld module analogues of standard elliptic curve schemes: 🦴 ($\mathbb{F}_q$ -linearity).

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Broader questions

Elliptic curves *vs* Drinfeld modules

Integers *vs* Polynomials

Number fields *vs* Function fields

Zero characteristic *vs* Positive characteristic

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Ore polynomials

Fix $K/\mathbb{F}_q$, and

$$\begin{aligned}\tau^n : \overline{K} &\rightarrow \overline{K} \\ x &\mapsto x^{q^n}.\end{aligned}$$

Definition of $K\{\tau\}$

Finite K -linear combinations of τ^n ; ring for addition and composition.

Properties

- Representation as polynomials: $K\{\tau\} = \{\sum_{i=0}^n x_i \tau^i, n \in \mathbb{Z}_{\geq 0}, x_i \in K\}$.
- Notion of τ -degree.
- Noncommutative: for $\lambda \in K$, $\tau^n \lambda = \lambda^{q^n} \tau^n$.
- Left-euclidean: for any $A, B \in K\{\tau\}$, there exist $Q, R \in K\{\tau\}$ such that:

$$A = QB + R, \quad \deg_{\tau}(R) < \deg_{\tau}(B).$$

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Representing Drinfeld modules

(Almost) Definition (Drinfeld, 1977)

A *Drinfeld $\mathbb{F}_q[T]$ -module over K* is a morphism of $\mathbb{F}_q[T]$ -algebras

$$\begin{aligned}\phi : \mathbb{F}_q[T] &\rightarrow K\{\tau\} \\ a &\mapsto \phi_a.\end{aligned}$$

Representation

ϕ is represented by ϕ_T . The *rank* of ϕ is $\deg_\tau(\phi_T)$.

Morphisms

A *morphism* $u : \phi \rightarrow \psi$ is an Ore polynomial $u \in K\{\tau\}$ such that

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The points of a Drinfeld module

For an elliptic curve, the *points* form a $\mathbb{Z}$ -module.

Geometric points

ϕ acts on $\overline{K}$ via

$$\begin{aligned}\mathbb{F}_q[T] \times \overline{K} &\rightarrow \overline{K} \\ (a, z) &\mapsto \phi_a(z).\end{aligned}$$

$\mathbb{F}_q[T]$ -module denoted by $\phi(\overline{K})$.

K -rational points

Write

$$\phi(K) := \phi(\overline{K}) \cap K.$$

The underlying set of $\phi(K)$ is always K .

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and

$$(\#E(\mathbb{F}_q)) \simeq (d_1 \cdots d_n)$$

Assume K is finite. Decompose

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The “number of K -rational points of ϕ ” ($\mathbb{F}_q[T]$ -cardinality) is

$$(|\phi(K)|) = (d_1 \cdots d_n).$$

Often referred to as the *Euler-Poincaré characteristic* or *Fitting ideal* of $\phi(K)$.

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First deterministic polynomial time: ~~birthday~~ Schoof, 1985.

Number of points *via* the Frobenius endomorphism

1. An elliptic curve $E/\mathbb{F}_q$ has a *Frobenius endomorphism* $\pi : (x, y) \mapsto (x^q, y^q)$.
2. π has a *characteristic polynomial*

$$\chi = X^2 - tX + q \in \mathbb{Z}[X]$$

such that

$$\chi(\pi) = \pi^2 - t\pi + q = 0.$$

3. We have

$$|E(\mathbb{F}_q)| = \chi(1).$$

Important invariant.

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1. Assume K is finite. A Drinfeld module ϕ over K has a *Frobenius endomorphism* $\pi = \tau^{[K:\mathbb{F}_q]} \in K\{\tau\}$.
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Abstract definition of χ

Tate module

For a prime $\mathfrak{q}$ distinct from $\mathfrak{p}$ and $n \geq 1$, the $\mathfrak{q}^n$ -torsion, denoted by $\phi[\mathfrak{q}^n]$, is isomorphic to $(\mathbb{F}_q[T]/\mathfrak{q}\mathbb{F}_q[T])^r$.

The $\mathfrak{q}$ -adic Tate module of ϕ is

$$T_{\mathfrak{q}}(\phi) = \varprojlim_{n \geq 1} \phi[\mathfrak{q}^n].$$

Definition of χ via Tate modules

The characteristic polynomial of the action of π on $T_{\mathfrak{q}}(\phi)$ has coefficients in A that do not depend on $\mathfrak{q}$.

Abstract definition of χ

Tate module

For a prime $\mathfrak{q}$ distinct from $\mathfrak{p}$ and $n \geq 1$, the $\mathfrak{q}^n$ -torsion, denoted by $\phi[\mathfrak{q}^n]$, is isomorphic to $(\mathbb{F}_q[T]/\mathfrak{q}\mathbb{F}_q[T])^r$.

The $\mathfrak{q}$ -adic Tate module of ϕ is

$$T_{\mathfrak{q}}(\phi) = \varprojlim_{n \geq 1} \phi[\mathfrak{q}^n].$$

Definition of χ via Tate modules

The characteristic polynomial of the action of π on $T_{\mathfrak{q}}(\phi)$ has coefficients in A that do not depend on $\mathfrak{q}$.

Anderson motives

Definition

$\mathbb{M}(\phi)$ is the $K[T]$ -module

$$\begin{aligned} K[T] \times K\{\tau\} &\rightarrow K\{\tau\} \\ (\sum_i \lambda_i T^i, f(\tau)) &\mapsto \sum_i \lambda_i f(\tau) \phi_T^i \end{aligned}$$

Canonical basis

$\mathbb{M}(\phi)$ is free with rank r (the rank of ϕ) with basis

$$(1, \tau, \dots, \tau^{r-1}).$$

Recursive process *via* Ore Euclidean division:

$$f(\tau) = Q(\tau)\phi_T + R(\tau), \quad \deg_{\tau}(R) < r.$$

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Morphisms as matrices

Any morphisms $u : \phi \rightarrow \psi$ gives a morphism on the Anderson motives

$$\begin{aligned} \mathbb{M}(u) : \mathbb{M}(\psi) &\rightarrow \mathbb{M}(\phi) \\ f &\mapsto fu. \end{aligned}$$

To compute the matrix of $\mathbb{M}(u)$, compute the coordinates of

$$f, \tau f, \dots, \tau^{r-1} f.$$

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Demo!

Our contribution (with Xavier Caruso)

Caruso, L., 2023

- Any endomorphism.
- Any r .
- Any K .
- Extends to isogeny norms.
- Any function ring.
- SageMath implementation in the standard library.

<i>2008</i>	Gekeler	Frobenius, $r = 2$ generalized to $r \in \mathbb{Z}_{\geq 0}$ by Musleh
<i>2019</i>	Musleh, Schost	Frobenius, $r = 2$
<i>2020</i>	Garai, Papikian	Frobenius, $r = 2$
<i>2023</i>	Musleh, Schost	Any endomorphism, any r
<i>2024</i>	Musleh	Any endomorphism, any r

Cost of computing χ

Las Vegas algorithm, cost in bit operations:

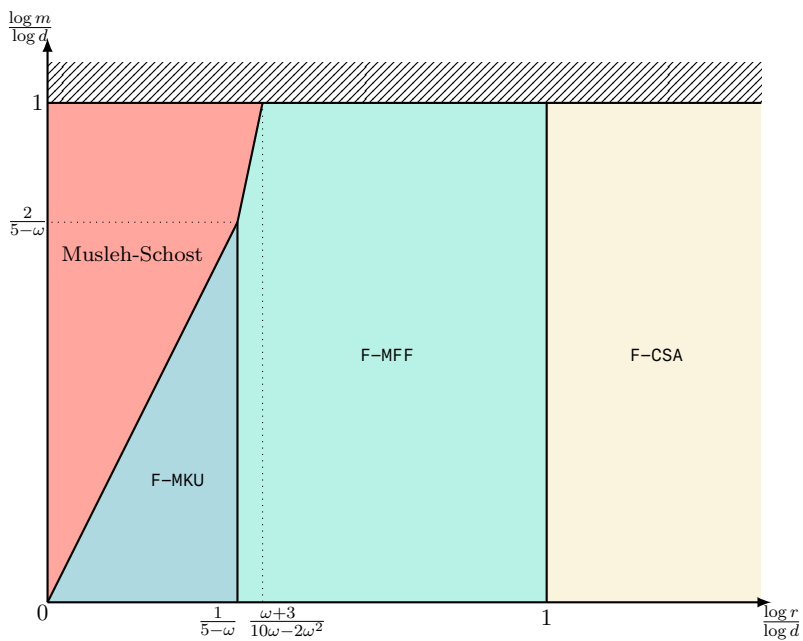
- [F-MFF] $O^{\sim}(d \log^2 q) + (\text{SM}^{\geq 1}(d, d) + d^2 r + d r^{\omega}) \log q)^{1+o(1)},$
- [F-MKU] $O^{\sim}(d \log^2 q) + ((d^2 r^{\omega-1} + d r^{\omega}) \log q)^{1+o(1)},$
- [F-CSA] $O^{\sim}(d \log^2 q) + (r d^{\omega} \log q)^{1+o(1)}.$

d = $[K : \mathbb{F}_q]$

r = rank of ϕ

ω = feasible exponent for matrix multiplication in a field

$\text{SM}^{\geq 1}$ = related to fast multiplication of Ore polynomials [Caruso-Le Borgne, 2017]



For general endomorphisms

Deterministic algorithm:

- ◦ $O^\sim(n^2 + (n + r)r^{\Omega-1})$ operations in K
- $O(n^2 + r^2)$ q -exponentiations in K

If K is finite, Las Vegas algorithm (cost in binary operations):

- $O^\sim(d \log^2 q) + ((\text{SM}^{\geq 1}(n, d) + ndr + (n + d)r^\omega) \log q)^{1+o(1)}.$

n = τ -degree of the endomorphism

d = $[K : \mathbb{F}_q]$

r = rank of ϕ

ω = feasible exponent for matrix multiplication in a field

Ω = feasible exponent for characteristic polynomial computation in a field

$\text{SM}^{\geq 1}$ = related to fast multiplication of Ore polynomials [Caruso-Le Borgne, 2017]

For isogeny norms

Deterministic algorithm:

- ○ $O^\sim(n^2 + nr^{\omega-1} + r^\omega)$ operations in K
- $O(n^2 + r^2)$ q -exponentiations in K

If K is finite, Las Vegas algorithm (cost in bit operations):

- $O^\sim(d \log^2 q) + ((\text{SM}^{\geq 1}(n, d) + ndr + n \min(d, r)r^{\omega-1} + dr^\omega) \log q)^{1+o(1)}.$

n = τ -degree of the isogeny

d = $[K : \mathbb{F}_q]$

r = rank of ϕ

ω = feasible exponent for matrix multiplication in a field

Ω = feasible exponent for characteristic polynomial computation in a field

$\text{SM}^{\geq 1}$ = related to fast multiplication of Ore polynomials [Caruso-Le Borgne, 2017]