Point counting without points

Antoine Leudière

University of Calgary

René 25

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The rules of point counting

Philosophy of Drinfeld modules

Representation of Drinfeld modules

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What is point counting?

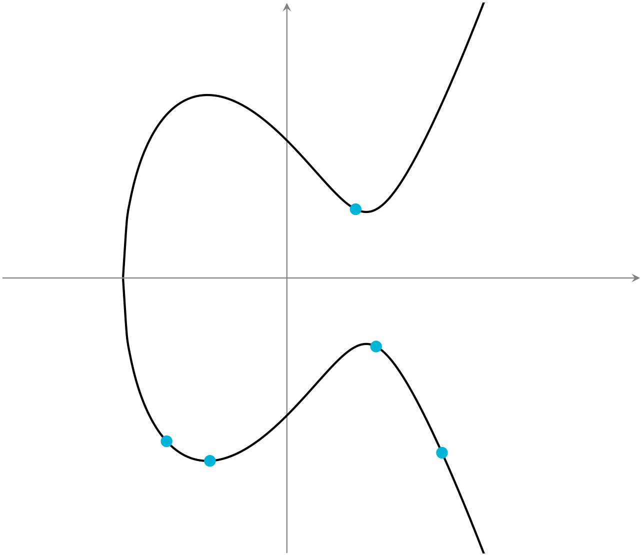
Naive approach

Counting solutions to an equation.

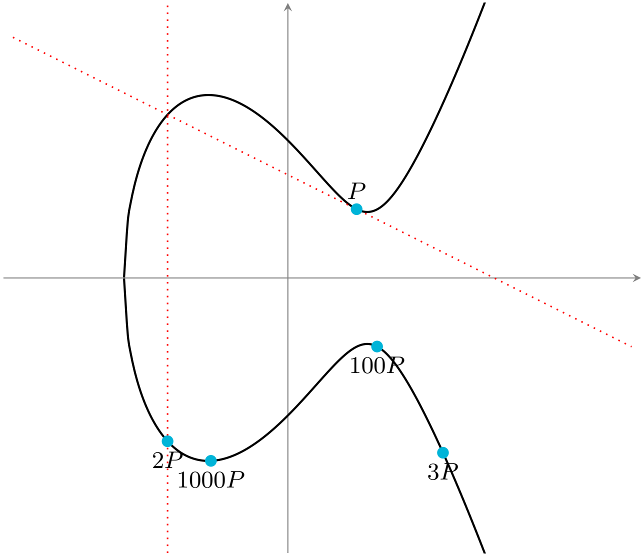
For algebraic varieties on a finite field: hard algorithmic problem.

Consider algebraic objects: *elliptic curves*, *abelian varieties*.

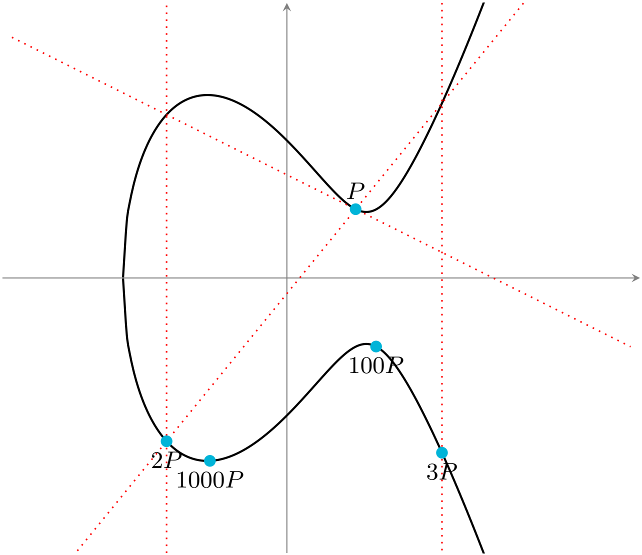
Algebraic structure



Algebraic structure



Algebraic structure



Changing the rules

Let E be an elliptic curve. As an abelian group,

$$E(\mathbb{F}_q) \simeq \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_n\mathbb{Z}.$$

So

$$\#E(\mathbb{F}_q) = |d_1 \cdots d_n|.$$

Let R be a PID, M be a finite R-module. There are $m_1, \ldots, m_\ell \in R$ s.t.:

$$M \simeq R/m_1R \times \cdots \times R/m_\ell R.$$

R-cardinality

Define the *R-cardinality of M* as

$$m_1 \cdots m_\ell.$$

An alternative to $\mathbb{Z}$

Consider replacing $\mathbb{Z}$ by $R = \mathbb{F}_q[T]$!

Both Euclidean rings:

- $\mathbb{Z}$: number fields;
- $\mathbb{F}_q[T]$: function fields.

Advantages of function fields

- Unconditional results (*e.g.* GRH).
- Faster algorithms (*e.g.* factorization).
- Geometrical properties of function fields.
- And others: $\mathbb{F}_q$-linearity, non-Archimedean analysis, etc.

What are elliptic curves for $R = \mathbb{F}_q[T]$?

Module structure	
$\mathbb{Z}$-module	$\mathbb{F}_q[T]$-module
Torsion	
$(\mathbb{Z}/n\mathbb{Z})^2$, $p \nmid n$	$(\mathbb{F}_q[T]/a\mathbb{F}_q[T])^2$, $\mathfrak{p} \nmid a$
Endomorphism ring	
$\mathbb{Z}$, order in $\mathbb{Q}(\sqrt{-d})$, order in $\mathcal{B}_{p,\infty}$	Same, over the function field $\mathbb{F}_q(T)$

What objects play give this? Drinfeld modules!

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Analogies

$\mathbb{Z}$	$\mathbb{F}_q[T]$
$\mathbb{Q}$	$\mathbb{F}_q(T)$
Number fields (finite ext.)	Function fields (finite ext.)
$\mathbb{R}$	$\mathbb{R}_\infty = \mathbb{F}_q((\frac{1}{T}))$
$\mathbb{C}$	$\mathbb{C}_\infty$ = completion of $\overline{\mathbb{R}_\infty}$
Elliptic curves	Drinfeld modules

Mantra

Our integers are polynomials.

Applications of Drinfeld modules

Introduced by Drinfeld (*elliptic modules*) in 1977. First works by Carlitz.

Function field arithmetics

- Explicit class field theory and theory of complex multiplication.
- Geometric Langlands program.
- Others: exponential and logarithm functions, Drinfeld modular forms, etc.

Computer algebra

State-of-the art factorization in $\mathbb{F}_q[T]$, by computing Hasse invariants (Doliskani-Narayanan-Schost, 2021).

Cryptography

Drinfeld module analogues of standard elliptic curve schemes: ☠️ ($\mathbb{F}_q$-linearity).

Highlights both similarities and fundamental differences with elliptic curves.

Broader questions

Elliptic curves	*vs*	Drinfeld modules
Integers	*vs*	Polynomials
Number fields	*vs*	Function fields
Zero characteristic	*vs*	Positive characteristic

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Ore polynomials

Fix $K/\mathbb{F}_q$, and

$$\begin{array}{rccc} \tau^n : & \overline{K} & \to & \overline{K} \\ & x & \mapsto & x^{q^n}. \end{array}$$

Definition of $K\{\tau\}$

Finite K-linear combinations of τ^n'; ring for addition and composition.

Properties

- Representation as polynomials: $K\{\tau\} = \{\sum_{i=0}^{n} x_i \tau^i, n \in \mathbb{Z}_{\geqslant 0}, x_i \in K\}$.
- Notion of τ-degree.
- Noncommutative: for $\lambda \in K$, $\tau^n \lambda = \lambda^{q^n} \tau^n$.
- Left-euclidean: for any $A, B \in K\{\tau\}$, there exist $Q, R \in K\{\tau\}$ such that:

$$A = QB + R, \quad \deg_\tau(R) < \deg_\tau(B).$$

Representing Drinfeld modules

(Almost) Definition (Drinfeld, 1977)

A *Drinfeld* $\mathbb{F}_q[T]$*-module over* K is a morphism of $\mathbb{F}_q[T]$-algebras

$$\begin{array}{rccc} \phi : & \mathbb{F}_q[T] & \to & K\{\tau\} \\ & a & \mapsto & \phi_a. \end{array}$$

Representation

ϕ is represented by ϕ_T. The *rank* of ϕ is $\deg_\tau(\phi_T)$.

Morphisms

A *morphism* $u : \phi \to \psi$ is an Ore polynomial $u \in K\{\tau\}$ such that

$$\forall a \in \mathbb{F}_q[T], \qquad u\phi_a = \psi_a u.$$

The points of a Drinfeld module

For an elliptic curve, the *points* form a $\mathbb{Z}$-module.

Geometric points

ϕ acts on $\overline{K}$ *via*

$$\begin{array}{rcl} \mathbb{F}_q[T] \times \overline{K} & \to & \overline{K} \\ (a, z) & \mapsto & \phi_a(z). \end{array}$$

$\mathbb{F}_q[T]$-module denoted by $\phi(\overline{K})$.

K-rational points

Write

$$\phi(K) := \phi(\overline{K}) \cap K.$$

The underlying set of $\phi(K)$ is always K.

The number of points

For an elliptic curve,

$$E(\mathbb{F}_q) \simeq \mathbb{Z}/(d_1) \times \cdots \times \mathbb{Z}/(d_n),$$

and

$$(\#E(\mathbb{F}_q)) \simeq (d_1 \cdots d_n)$$

Assume K is finite. Decompose

$$\phi(K) \simeq \mathbb{F}_q[T]/(d_1) \times \cdots \times \mathbb{F}_q[T]/(d_n).$$

The "number of K-rational points of ϕ" ($\mathbb{F}_q[T]$-cardinality) is

$$(|\phi(K)|) = (d_1 \cdots d_n).$$

Often referred to as the *Euler-Poincaré characteristic* or *Fitting ideal* of $\phi(K)$.

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The elliptic curve case

First deterministic polynomial time: ~~birthday boy~~ Schoof, 1985.

Number of points *via* the Frobenius endomorphism

1. An elliptic curve $E/\mathbb{F}_q$ has a *Frobenius endomorphism* $\pi : (x, y) \mapsto (x^q, y^q)$.
2. π has a *characteristic polynomial*

$$\chi = X^2 - tX + q \in \mathbb{Z}[X]$$

   such that

$$\chi(\pi) = \pi^2 - t\pi + q = 0.$$

3. We have

$$|E(\mathbb{F}_q)| = \chi(1).$$

Important invariant.

The Drinfeld module case

1. Assume K is finite. A Drinfeld module ϕ over K has a *Frobenius endomorphism* $\pi = \tau^{[K:\mathbb{F}_q]} \in K\{\tau\}$.
2. π has a *characteristic polynomial*

$$\chi = X^r + a_{r-1}(T)X^{r-1} + \cdots + a_1(T)X + a_0(T) \in \mathbb{F}_q[T][X]$$

   such that

$$\chi(\pi) = \pi^r + \phi_{a_{r-1}}\pi^{r-1} + \cdots + \phi_{a_1}\pi + \phi_{a_0} = 0.$$

3. We have (Gekeler, 1991)

$$(|\phi(K)|) = (\chi(1))$$

Important invariant.

Abstract definition of χ

Tate module

For a prime $\mathfrak{q}$ distinct from $\mathfrak{p}$ and $n \geqslant 1$, the $\mathfrak{q}^n$-torsion, denoted by $\phi[\mathfrak{q}^n]$, is isomorphic to $(\mathbb{F}_q[T]/\mathfrak{q}\mathbb{F}_q[T])^r$.
The $\mathfrak{q}$-*adic Tate module* of ϕ is

$$T_{\mathfrak{q}}(\phi) = \varprojlim_{n \geqslant 1} \phi[\mathfrak{q}^n].$$

Definition of χ *via* Tate modules

The characteristic polynomial of the action of π on $T_{\mathfrak{q}}(\phi)$ has coefficients in A that do not depend on $\mathfrak{q}$.

Anderson motives

Definition

$\mathbb{M}(\phi)$ is the $K[T]$-module

$$\begin{array}{rcl} K[T] \times K\{\tau\} & \to & K\{\tau\} \\ \left(\sum_i \lambda_i T^i, f(\tau)\right) & \mapsto & \sum_i \lambda_i f(\tau)\phi_T^i \end{array}$$

Canonical basis

$\mathbb{M}(\phi)$ is free with rank r (the rank of ϕ) with basis

$$(1, \tau, \ldots, \tau^{r-1}).$$

Recursive process *via* Ore Euclidean division:

$$f(\tau) = Q(\tau)\phi_T + R(\tau), \quad \deg_\tau(R) < r.$$

Morphisms as matrices

Any morphisms $u : \phi \to \psi$ gives a morphism on the Anderson motives

$$\begin{array}{rrcl} \mathbb{M}(u): & \mathbb{M}(\psi) & \to & \mathbb{M}(\phi) \\ & f & \mapsto & fu. \end{array}$$

To compute the matrix of $\mathbb{M}(u)$, compute the coordinates of

$$f, \tau f, \cdots, \tau^{r-1} f.$$

Demo!

Our contribution (with Xavier Caruso)

Caruso, L., 2023

- Any endomorphism.
- Any r.
- Any K.
- Extends to isogeny norms.
- Any function ring.
- SageMath implementation in the standard library.

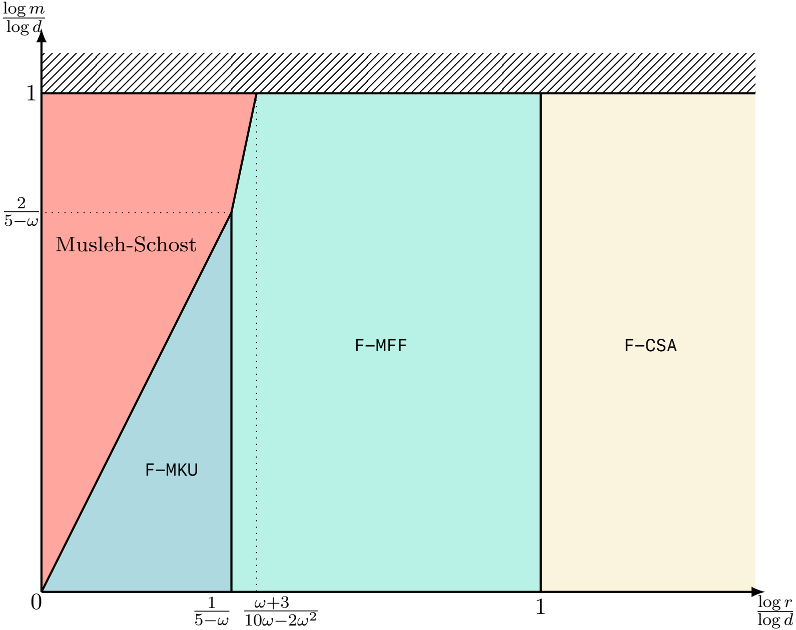
2008	Gekeler	Frobenius, $r = 2$ generalized to $r \in \mathbb{Z}_{\geqslant 0}$ by Musleh
2019	Musleh, Schost	Frobenius, $r = 2$
2020	Garai, Papikian	Frobenius, $r = 2$
2023	Musleh, Schost	Any endomorphism, any r
2024	Musleh	Any endomorphism, any r

Cost of computing χ

Las Vegas algorithm, cost in bit operations:

- [F−MFF] $O\tilde{}(d\log^2 q) + (\mathrm{SM}^{\geqslant 1}(d,d) + d^2 r + d r^\omega)\log q)^{1+o(1)}$,
- [F−MKU] $O\tilde{}(d\log^2 q) + ((d^2 r^{\omega-1} + d r^\omega)\log q)^{1+o(1)}$,
- [F−CSA] $O\tilde{}(d\log^2 q) + (r d^\omega \log q)^{1+o(1)}$.

d	$=$	$[K : \mathbb{F}_q]$
r	$=$	rank of ϕ
ω	$=$	feasible exponent for matrix multiplication in a field
$\mathrm{SM}^{\geqslant 1}$	$=$	related to fast multiplication of Ore polynomials [Caruso-Le Borgne, 2017]


$\frac{\log m}{\log d}$
1
$\frac{2}{5-\omega}$
Musleh-Schost
F−MKU
F−MFF
F−CSA
0
$\frac{1}{5-\omega}$
$\frac{\omega+3}{10\omega-2\omega^2}$
1
$\frac{\log r}{\log d}$

For general endomorphisms

Deterministic algorithm:

- ◦ $O\tilde{}(n^2 + (n+r)r^{\Omega-1})$ operations in K
  - ◦ $O(n^2 + r^2)$ q-exponentiations in K

If K is finite, Las Vegas algorithm (cost in binary operations):

- $O\tilde{}(d \log^2 q) + ((\mathrm{SM}^{\geqslant 1}(n,d) + ndr + (n+d)r^{\omega}) \log q)^{1+o(1)}$.

n	=	τ-degree of the endomorphism
d	=	$[K : \mathbb{F}_q]$
r	=	rank of ϕ
ω	=	feasible exponent for matrix multiplication in a field
Ω	=	feasible exponent for characteristic polynomial computation in a field
$\mathrm{SM}^{\geqslant 1}$	=	related to fast multiplication of Ore polynomials [Caruso-Le Borgne, 2017]

For isogeny norms

Deterministic algorithm:

- ◦ $\tilde{O}(n^2 + nr^{\omega-1} + r^{\omega})$ operations in K
  - ◦ $O(n^2 + r^2)$ q-exponentiations in K

If K is finite, Las Vegas algorithm (cost in bit operations):

- $\tilde{O}(d \log^2 q) + ((\mathrm{SM}^{\geqslant 1}(n, d) + ndr + n \min(d, r) r^{\omega-1} + dr^{\omega}) \log q)^{1+o(1)}$.

n	=	τ-degree of the isogeny
d	=	$[K : \mathbb{F}_q]$
r	=	rank of ϕ
ω	=	feasible exponent for matrix multiplication in a field
Ω	=	feasible exponent for characteristic polynomial computation in a field
$\mathrm{SM}^{\geqslant 1}$	=	related to fast multiplication of Ore polynomials [Caruso-Le Borgne, 2017]