

# A computation on Drinfeld modules

Simon Fraser University

*Number Theory & Algebraic Geometry Seminar*

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May 29th, 2025

Introduction

Ore polynomials

Drinfeld modules

Computing a group action

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# Kronecker-Weber theorem

Every abelian number field lies in a cyclotomic field  $\mathbb{Q}_n$ .

$\mathbb{Q}_n$  is generated by the  $n$ -th roots of unity.

Alternative construction

Consider the  $\mathbb{Z}$ -module

$$\begin{array}{ccc} \mathbb{Z} \times \overline{\mathbb{Q}}^* & \rightarrow & \overline{\mathbb{Q}}^* \\ (n, z) & \mapsto & z^n \end{array}$$

The  $n$ -th roots of unity are the  $n$ -torsion of this  $\mathbb{Z}$ -module.

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# Class Field Theory

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Given a number field  $K/\mathbb{Q}$ , what can I say about the abelian extensions of  $K$ , using only objects defined in  $K$ ?

Very important object

*Hilbert class field* (maximal unramified abelian extension).

Some explicit results:

- Kronecker-Weber.
- The case of quadratic imaginary number fields ( $\mathbb{Q}(\sqrt{-d})$ , where  $d < 0$ ).

The Hilbert class field of  $\mathbb{Q}(\sqrt{-d})$  is generated by isomorphism classes of *elliptic curves* with complex multiplication in  $\mathbb{Q}(\sqrt{-d})$ .

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# An alternative framework

Common point between the results:

- Number fields (characteristic 0).
- $\mathbb{Z}$ -modules.

Can we change these?

Zero characteristic	Positive characteristic
$\mathbb{Z}$	$\mathbb{F}_q[T]$
$\mathbb{Q}$	$\mathbb{F}_q(T)$
Number fields (finite ext.)	Function fields (finite ext.)
$\mathbb{R}$	$\mathbb{R}_\infty = \mathbb{F}_q((1/T))$
$\mathbb{C}$	$\mathbb{C}_\infty = \text{completion of } \overline{\mathbb{R}_\infty}$
Roots of unity	Drinfeld modules
Elliptic curves	Drinfeld modules

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# Advantages of function fields

- Geometrical interpretation.
- Non-Archimedean valuations.
- Faster algorithms (polynomial derivation and factorization).
- Some unconditional results (GRH).

# Broader questions

Elliptic curves    *vs*    Drinfeld modules

Integers    *vs*    Polynomials

Number fields    *vs*    Function fields

Zero characteristic    *vs*    Positive characteristic



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# Ore polynomials $K\{\tau\}$

Consider an extension  $K/\mathbb{F}_q$  and the Frobenius endomorphisms

$$\begin{aligned}\tau^n : \overline{K} &\rightarrow \overline{K} \\ x &\mapsto x^{q^n}.\end{aligned}$$

Finite  $K$ -linear combinations of  $\tau^n$ : ring  $K\{\tau\}$  for addition and composition.

## Properties

- Representation as polynomials:  $K\{\tau\} = \{\sum_{i=0}^n x_i \tau^i, n \in \mathbb{Z}_{\geq 0}, x_i \in K\}$ .
- Notion of  $\tau$ -degree.
- Noncommutative: for  $\lambda \in K$ ,  $\tau^n \lambda = \lambda^{q^n} \tau^n$ .

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# Euclidean divisions

$K\{\tau\}$  is left-euclidean

For all  $A(\tau), B(\tau) \in K\{\tau\}$ , there exist  $Q(\tau), R(\tau) \in K\{\tau\}$  such that:

$$\begin{cases} A(\tau) = Q(\tau)B(\tau) + R(\tau), \\ \deg_{\tau}(R(\tau)) < \deg_{\tau}(B(\tau)). \end{cases}$$

# Kernels and Ore polynomials

A bijection

$$\left\{ \begin{array}{l} \text{Ore polynomials } f \in K\{\tau\} \\ \text{with constant term 1} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{finite dimensional } \mathbb{F}_q\text{-linear subspaces} \\ V \subset K^{\text{sep}} \text{ stable by } \text{Gal}(K^{\text{sep}}/K) \end{array} \right\}$$
$$f \mapsto \ker f.$$

$$\begin{array}{lll} f_1 & \longleftrightarrow & V_1 \\ f_2 & \longleftrightarrow & V_2 \\ \text{rgcd}(f_1, f_2) & \longleftrightarrow & V_1 \cap V_2 \end{array}$$

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# Representing Drinfeld modules

Drinfeld modules  $\phi$  and their morphisms are represented as Ore polynomials.

## Representation

For  $a \in \mathbb{F}_q[T]$ , the endomorphism of multiplication by  $a$  is represented by an Ore polynomial  $\phi_a \in K\{\tau\}$ .

From now on,  $K$  is finite with  $[K : \mathbb{F}_q] = d$ .

## Frobenius endomorphism

One extra endomorphism:  $\text{Frob} = \tau^d \in K\{\tau\}$ .

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# The module of a Drinfeld module

A Drinfeld module is *not* a module!

Let  $\phi$  be a Drinfeld module.

We have an  $\mathbb{F}_q[T]$ -module law on  $\overline{K}$ :

$$\begin{aligned}\mathbb{F}_q[T] \times \overline{K} &\rightarrow \overline{K} \\ (a, x) &\mapsto \phi_a(x).\end{aligned}$$

Drinfeld module analogue of the  $\mathbb{Z}$ -module coming from an elliptic curve!

The notion of *point* is ambiguous.

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Joint-work with P.-J. Spaenlehauer.

*Computing a group action from the class field theory of imaginary hyperelliptic function fields.*

Journal of symbolic computation, 2024.

<https://doi.org/10.1016/j.jsc.2024.102311>.

# Cryptography with group actions

Fix and assume:

- A free-transitive action of an abelian group  $G$  on a set  $X$ .
- For all  $g \in G$ , computing  $g$  from  $x$  and  $g \cdot x$  is hard.



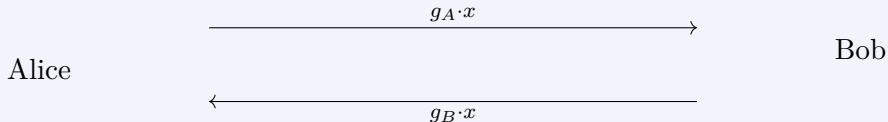
Hard Homogeneous Spaces (Couveignes, 1997)

Alice and Bob can use  $(g_A g_B) \cdot x = g_A \cdot (g_B \cdot x) = g_B \cdot (g_A \cdot x)$  as their secret key.

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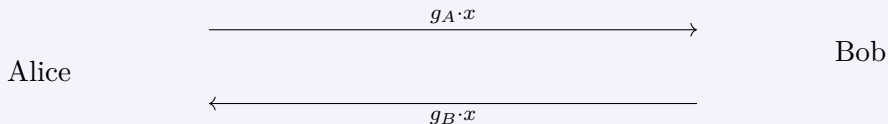
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# Isogeny-based cryptography

- Quadratic imaginary number field  $\mathbb{Q}(\sqrt{-d})$ .
- Its ring of integers  $\mathcal{O}$ , and the class group  $\text{Cl}(\mathcal{O})$ .
- The set  $X_{\mathcal{O}}$  isomorphism classes of elliptic curves with endomorphism ring  $\mathcal{O}$ .

Fix an ideal  $\mathfrak{a} \subset \mathcal{O}$  and an elliptic curve  $E$ . There is a curve  $E_{\mathfrak{a}}$  and morphism  $E \rightarrow E_{\mathfrak{a}}$  with kernel

$$\bigcap_{f \in \mathfrak{a}} \ker f.$$

We define

$$\overline{\mathfrak{a}} * \overline{E} = \overline{E_{\mathfrak{a}}}.$$

Free and transitive group action of  $\text{Cl}(\mathcal{O})$  on  $X_{\mathcal{O}}$ .

Too slow for cryptography (de Feo-Kieffer-Smith, 2018)!

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# The Drinfeld module analogue

Same action for Drinfeld modules!

- Order  $\mathcal{O}$  in an imaginary quadratic **number** field.
  - **Elliptic curves**  $E$  such that  $\text{End}(E) \simeq \mathcal{O}$ .
  - Ideal  $\mathfrak{a}$  of  $\text{End}(E)$ .
  - Compute  $\bigcap_{f \in \mathfrak{a}} \ker f$ .
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- Order  $\mathcal{O}$  in an imaginary quadratic **function** field.
  - **Drinfeld modules**  $\phi$  such that  $\text{End}(\phi) \simeq \mathcal{O}$ .
  - Ideal  $\mathfrak{a}$  of  $\text{End}(\phi)$ .
  - Compute  $\bigcap_{f \in \mathfrak{a}} \ker f$ .

# The hyperelliptic case

*Imaginary hyperelliptic curve*  $\mathcal{H}$  defined by  $\chi \in \mathbb{F}_q[T][X]$ , and its *coordinate ring*

$$\mathbb{F}_q[\mathcal{H}] = \mathbb{F}_q[T][X]/(\chi).$$

Mumford coordinates

Elements of  $\text{Cl}(\mathbb{F}_q[\mathcal{H}])$  are represented by couples  $(u, v) \in \mathbb{F}_q[T]^2$  with:

$$(u, v) \longleftrightarrow (\overline{u(T)}, \overline{X - v(T)})$$

For a Drinfeld module  $\phi$  such that  $\text{End}(\phi) = \mathbb{F}_q[\mathcal{H}]$ , we have an isomorphism

$$\begin{aligned} \mathbb{F}_q[\mathcal{H}] &\rightarrow \text{End}(\phi) \\ P(\textcolor{red}{T}, \textcolor{red}{X}) &\mapsto P(\phi_{\textcolor{red}{T}}, \textcolor{red}{\text{Frob}}). \end{aligned}$$

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# Practical computation

We rely on:

- Mumford coordinates.
- The correspondence between kernels and Ore polynomials.

Computing the action essentially goes down to computing

$$\text{rgcd}(\phi_u, \text{Frob} - \phi_v).$$

Computation time on cryptographic sizes goes from  $\sim 10$  min. to 400 ms.  
Highly insecure though! (Wesolowski)

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# Conclusive remarks

## Applications of Drinfeld modules

- Geometric Langlands program, Class Field Theory of function fields, GRH for function fields.
- State of the art polynomial factorization (Doliskani-Narayanan-Schost, 2018).

## Tools for Drinfeld modules

- Inspiration from elliptic curves.
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