Drinfeld modules in SageMath

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Outline of the Talk

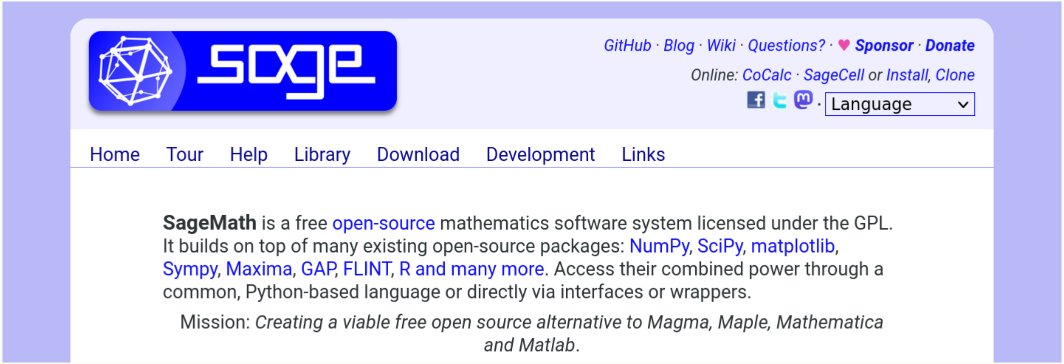
What is SageMath?

What is a Drinfeld module?

What are Drinfeld modules in SageMath?

History

SageMath is the leading computer algebra FOSS system. It was created in 2005 by William Stein; hundreds of mathematicians contributed to it.



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Context

Drinfeld modules were introduced by Vladimir Drinfeld in the 1970s to solve problems from the *class field theory of function fields*.

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😍 Growing interest for computational research on Drinfeld modules.

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Definition

Let:

- $\mathbb{F}_q$ be a finite field with q elements.
- K be a field containing $\mathbb{F}_q$.
- $\mathbb{F}_q[T]$ be the ring of polynomials with coefficients in $\mathbb{F}_q$.
- $K\{\tau\}$ be the ring of "skew" polynomials $a_0 + a_1\tau + \cdots + a_n\tau^n$ in K satisfying $\tau a_i = a_i^q \tau$, for all $a_i \in K$:

$$K\{\tau\} = \left\{ \sum_{i=0}^{n} a_i \tau^i, \quad n \in \mathbb{Z}_{\geqslant 0}, a_i \in K \right\}.$$

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A Drinfeld module is a special case of $\mathbb{F}_q$-algebra morphism

$$\phi : \mathbb{F}_q[T] \to K\{\tau\}.$$

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Representation

A Drinfeld module $\phi : \mathbb{F}_q[T] \to K\{\tau\}$ can be represented by:

- A morphism.
- A skew polynomial $\phi(T) = g_0 + g_1\tau + \cdots + g_r\tau^r$.
- A list of coefficients $[g_0, g_1, \ldots, g_r]$.

❌ A Drinfeld module is *not* a set!

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What are Drinfeld modules in SageMath?

The Parent/Element framework

SageMath is built on the Parent/Element framework: SageMath objects are usually either a set (`Parent`) or an element in the set (`Element`). And `Parents` should belong to a category. This does not really fit Drinfeld modules:

- Drinfeld form a category, and as such should be `Parents`.
- But Drinfeld modules have no underlying sets, so they don't have elements and as such should not be `Parents`.
- A Drinfeld module is a special kind of morphism, so it *technically* is an element in a set of morphisms, but mathematicians do not think about them this way.

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Possible solutions

There are multiple possible solutions:

1. Making Drinfeld modules `Parents` without `Elements`. In fact this solution has a strong mathematical soundness. Drawbacks: `Parents` are supposed to have elements; their category must be a subcategory of the category of sets.
2. Making Drinfeld modules a `CategoryObject`. Drawbacks: this class is barely used in the codebase.
3. Making Drinfeld modules elements and their category a Parent without a category. Drawbacks: no mathematical satisfaction, and this prevents from having a standard implementation for morphisms.

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