Drinfeld modules

Effective class group action and implementation

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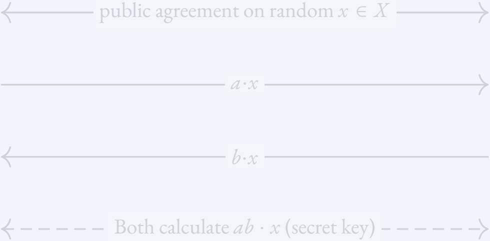
Institut de recherche mathématique de Rennes, *Géométrie et algèbre effectives* seminar

2022 september 23rd

Post quantum key-exchange and signature (1/2)

The late queen and duke choose an abelian simply transitive group action $G \times X \to X$.



public agreement on random $x \in X$

$a \cdot x$

$b \cdot x$

Both calculate $ab \cdot x$ (secret key)

Definition (Couveignes, 1996)

If computing $ab \cdot x$ knowing x, $a \cdot x$, $b \cdot x$ is hard, this is a hard homogeneous space.

Beullens-Kleinjung-Vercauteren in CSI-FiSh

Knowing the group order, we can build efficient signature schemes.

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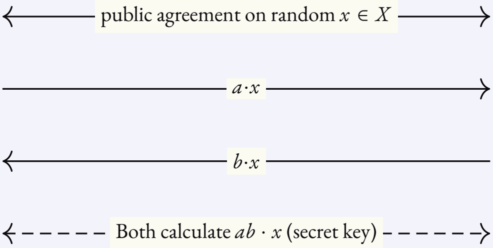

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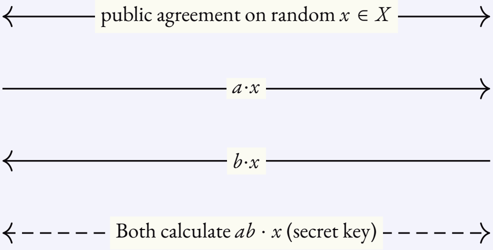

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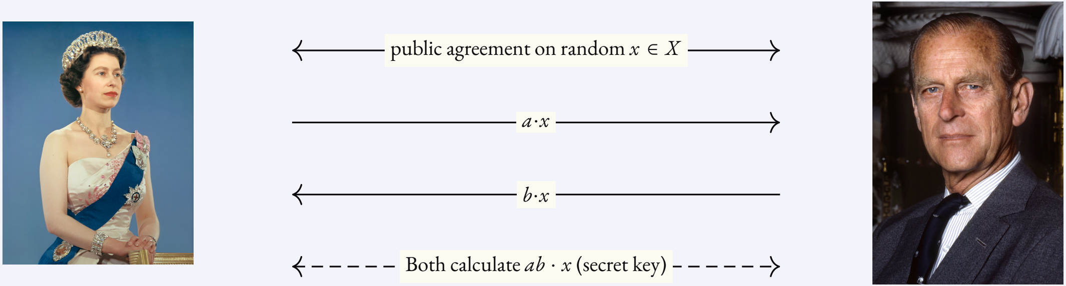

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Post quantum key-exchange and signature (2/2)

Diffie-Hellman ('76)	$G = \mathbb{Z}/n\mathbb{Z}$ X = cyclic group with order n and generator g Quantum-broken
CRS ('96, '04)	$G = \mathrm{Cl}(\mathbb{Q}(\sqrt{-D}))$ X = subset of isomorphism classes of ordinary elliptic curves. Slow to run & hard to know group order
CSIDH ('18)	$G = \mathrm{Cl}(\mathbb{Q}(\sqrt{-D}))$ X = subset of isomorphism classes of supersingular elliptic curves. Hard to know group order

Our hope

- Build a fast "Drinfeld analogue" of the CRS group action.
- Practical computation of the group order using Kedlaya's algorithm.

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Drinfeld modules make explicit the class field theory of function fields.
They play the role of elliptic curves for building the Hilbert class field of a function field.

Rule of thumb

$$\text{Elliptic curves} \xleftrightarrow{\text{behave like}} \text{Drinfeld modules with rank two.}$$

Algorithms

- Ore polynomials: Caruso-Leborgne.
- Characteristic polynomial of the Frobenius endomorphism: Schost-Musleh, 2019.
- Modular polynomials of rank 2 Drinfeld modules: Caranay-Greenberg-Scheidler, 2019.
- Tools for isogenies and endomorphisms: Caranay's thesis, 2018; Caranay-Greenberg-Scheidler, 2019; Wesolowski, 2022.
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Drinfeld modules and elliptic curves

Number fields	**Function fields**
Base ring: $\mathbb{Z}$	Base ring: $\mathbb{F}_q[X]$
Fraction field: $\mathbb{Q}$	Fraction field: $\mathbb{F}_q(X)$
Finite extensions: number fields	Finite extensions: function fields

Elliptic curves	**Drinfeld $\mathbb{F}_q[X]$-modules, rank 2**
$\mathbb{Z}$-module law on $E(K)$	$\mathbb{F}_q[X]$-module law on K
Vélu formulae	
j-invariant encoding $\overline{\mathbb{F}_q}$-isomorphism classes	
Theory of complex multiplication	
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Main results [arXiv:2203.06970]

Computer algebra

- Definition & proof of a simply transitive CRS-like group action for Drinfeld modules.
- Efficient algorithm to compute the action.
- Efficient C++/NTL implementation.

Cryptography

- Reduction of the inverse problem to the isogeny-finding problem.
- Conjecture that the best (at the time) algorithm ran in exponential time.
- Wesolowski since found a polynomial algorithm (`ia.cr/2022/438`).

Software

- SageMath implementation from scratch of Drinfeld modules.
- To be integrated in SageMath.

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Let's find the definition

Let $K/\mathbb{F}_q$ be a field extension with a ring morphism

$$\gamma : \mathbb{F}_q[X] \to K.$$

Fact: a Drinfeld modules ϕ induces an $\mathbb{F}_q[X]$-module structure on K.
Let's find the definition from there!

Let $a, b \in \mathbb{F}_q[X], x, y \in K, \lambda \in \mathbb{F}_q$.

$$\begin{aligned} (1) \quad & a \cdot (x + y) = a \cdot x + a \cdot y; \\ (2) \quad & \lambda \cdot x = \lambda x; \\ (1) + (2) \Rightarrow \quad & \text{the map } \phi_a : x \mapsto a \cdot x \text{ is in } \mathrm{End}_{\mathbb{F}_q}(K). \\ (3) \quad & a \cdot (b \cdot x) = (ab) \cdot x; \\ (1) + (2) + (3) \Rightarrow \quad & \text{the map } a \mapsto \phi_a \text{ is a ring morphism } \mathbb{F}_q[X] \to \mathrm{End}_{\mathbb{F}_q}(K). \end{aligned}$$

We *will* define a Drinfeld module as a morphism $\mathbb{F}_q[X] \to \mathrm{End}_{\mathbb{F}_q}(K)$ with extra properties.

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(1) $a \cdot (x + y) = a \cdot x + a \cdot y$;

(2) $\lambda \cdot x = \lambda x$;

(1) + (2) ⇒ the map $\phi_a : x \mapsto a \cdot x$ is in $\mathrm{End}_{\mathbb{F}_q}(K)$.

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We *will* define a Drinfeld module as a morphism $\mathbb{F}_q[X] \to \mathrm{End}_{\mathbb{F}_q}(K)$ with extra properties.

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Let $K/\mathbb{F}_q$ be a field extension with a ring morphism

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Endomorphisms are Ore polynomials

$$\mathrm{End}_{\mathbb{F}_q}(K) = K\{\tau\} = \left\{\sum_{i=1}^{n} x_i \tau^n : n \geqslant 0, x_i \in K, \tau : x \mapsto x^q\right\}.$$

This is the ring of Ore polynomials; multiplication is endomorphism composition.

- Non-commutative polynomials: $\forall a \in K, \tau a = a^q \tau$.
- Left-Euclidean domain for the τ-degree.
- SageMath implementation by Caruso.

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Definition

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A Drinfeld module over γ is an $\mathbb{F}_q$-algebra morphism

$$\begin{aligned} \phi : \mathbb{F}_q[X] &\longrightarrow K\{\tau\} \\ P &\longmapsto \phi_P \end{aligned}$$

such that

$$\phi_X = a_0 + \cdots + a_r \tau^r$$

and $r > 0$, $a_0 = \gamma(X)$.

Module law

We define an $\mathbb{F}_q[X]$-module on K:

$$\begin{aligned} \mathbb{F}_q[X] \times K &\longrightarrow K \\ (P, z) &\longmapsto \phi_P(z). \end{aligned}$$

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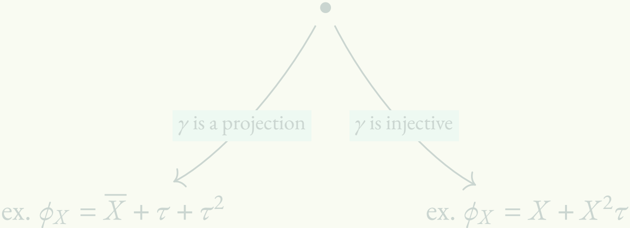
Example

In our case, a Drinfeld module is uniquely defined by ϕ_X.

Example

Two main situations for the base morphism $\gamma : \mathbb{F}_q[X] \to K$:



γ is a projection

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ex. $\phi_X = \overline{X} + \tau + \tau^2$

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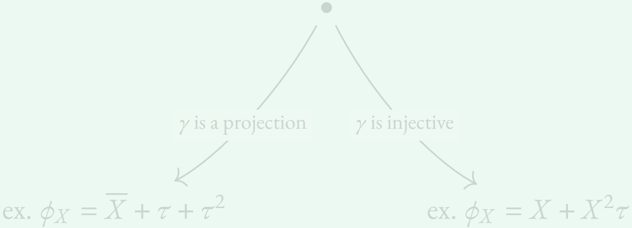

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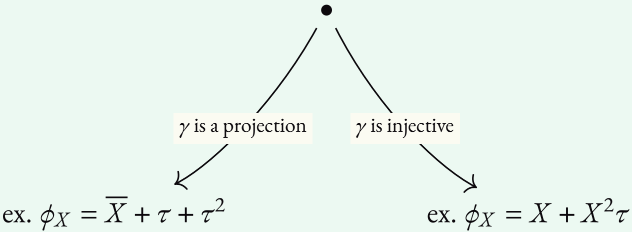

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Morphisms, isogenies

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A morphism of Drinfeld modules $\phi \longrightarrow \psi$ is an Ore polynomial $u \in K\{\tau\}$ such that

$$u\phi_P = \psi_P u, \quad \forall P \in \mathbb{F}_q[X],$$

i.e.

$$u\phi_X = \psi_X u.$$

An isogeny is a non-zero morphism.

Example

- $\phi_P \in \mathrm{End}(\phi)$ for all $P \in \mathbb{F}_q[X]$, i.e. $\mathbb{F}_q[X] \subset \mathrm{End}(\phi)$.
- $\mathbb{F}_q = \mathbb{F}_2, K = \mathbb{F}_2(i), \phi_X = i + i\tau + \tau^2, \psi_X = i + (i+1)\tau + \tau^2$ and $u = i + t$. Then $(i+t)(i+i\tau+\tau^2) = (i+t)(i+(i+1)\tau+\tau^2)$ and u is an isogeny $\phi \longrightarrow \psi$.

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Complex multiplication 1/2

Further hypotheses

- γ is surjective (ergo K is finite).
- $\mathrm{rank}(\phi) := \deg_\tau(\phi_X) = 2$.

Definition

Define the Frobenius endomorphism τ_K of ϕ as

$$\tau_K : x \mapsto x^{\#K}.$$

Theorem (Schost-Musleh)

There exists $\chi \in \mathbb{F}_q[X][T]$, *called the polynomial characteristic of the Frobenius endomorphism, such that*

$$\chi(\phi_X)(\tau_K) = 0$$

and $\chi(X)(T) = T^2 - A(X)T + B(X)$ *and* $\deg(A) \leqslant [K : \mathbb{F}_q]$, $\deg(B) \leqslant \deg(A)/2$.

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ϕ is ordinary if the Frobenius trace (middle coefficient of χ) is not in $\mathrm{Ker}(\gamma)$.

The characteristic polynomial χ can be efficiently computed: Schost-Musleh, 2019.

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Theorem (L.-Spaenlehauer, 2022)

The class group of $\mathrm{End}(\phi)$ *acts freely and transitively on the set* S *of isomorphism classes of rank two Drinfeld module that are isogeneous to* ϕ*.*

Let $I \subset \mathrm{End}(\phi)$ be an ideal and ψ be a rank two Drinfeld module.
There exists (Vélu formulae) an isogeny with domain ψ whose kernel is

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We map (I, ψ) to its codomain.

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The action is defined as the extension to class group and isomorphism classes of this map.

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Representation of the class group (1/2)

$$\mathbb{F}_q[X][T]/(\chi) \simeq \mathrm{End}(\phi) \simeq \{f \in \mathbb{F}_q(\mathcal{H}) : f \text{ regular everywhere outside } \infty\}.$$

Elements of $\mathrm{Pic}^0(\mathcal{H})$ are represented by Mumford coordinates: couples $(u, v) \in \mathbb{F}_q[X]^2$ verifying:

- u is monic;
- $\deg(v) < \deg(u) \leqslant ([K : \mathbb{F}_q] - 1)/2$;
- $u \mid \chi(X, v(X))$.

$$\begin{aligned} \mathrm{Pic}^0(\mathcal{H}) &\xrightarrow{\simeq} \mathrm{Cl}\left(\mathbb{F}_q[X][T]/(\chi)\right) \\ (u, v) &\mapsto \text{class of } \langle u(X), T - v(X)\rangle, \end{aligned}$$

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$$\bigcap_{f \in I} \mathrm{Ker}(f) = \bigcap_{\bar{f} \in \text{ideal of } \mathbb{F}_q[X][T]/(\chi)} \mathrm{Ker}(f(\phi_X, \tau_K))$$

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$$= \mathrm{Ker}(\phi_u) \cap \mathrm{Ker}(\tau_K - \phi_v)$$

The isogeny corresponding to this kernel (Vélu formula) is

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Input: — Mumford coordinates $(u, v) \in \mathbb{F}_q[X]^2$.
       — A j-invariant $j \in K$.
Output: A $j$-invariant.
$\widetilde{u} \leftarrow u(j^{-1}\tau^2 + \tau + \gamma(X)) \in K\{\tau\}$;
$\widetilde{v} \leftarrow v(j^{-1}\tau^2 + \tau + \gamma(X)) \in K\{\tau\}$;
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$\widehat{g} \leftarrow \iota_0^{-q}(\iota_0 + \iota_1(\gamma(X)^q - \gamma(X)))$;
$\widehat{\Delta} \leftarrow j^{-q^{\deg_\tau(\iota)}}$;
Return $\widehat{g}^{q+1}/\widehat{\Delta}$.
```

C++ / NTL implementation with crypto parameters: ~200 ms computation for $\mathbb{F}_q = \mathbb{F}_2$, $K = \mathbb{F}_{2^{521}}$, $\mathrm{genus}(\mathcal{H}) = 260$ and a Jacobian with order

$$2 \times 315413182467545672604116316415047743350494962889744865259442943656024073295689.$$

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$\widehat{\Delta} \leftarrow j^{-q^{\deg_\tau(\iota)}}$;
Return $\widehat{g}^{q+1}/\widehat{\Delta}$.
```

C++ / NTL implementation with crypto parameters: ~200 ms computation for $\mathbb{F}_q = \mathbb{F}_2$, $K = \mathbb{F}_{2^{521}}$, $\mathrm{genus}(\mathcal{H}) = 260$ and a Jacobian with order

$$2 \times 315413182467545672604116316415047743350494962889744865259442943656024073295689.$$

Back to crypto

It's fast. But is it safe?

No.

Security relies on the hardness of finding a fixed-degree isogeny between two Drinfeld modules.
Write $\phi_X = \Delta\tau^2 + g\tau + \omega$, $\psi_X = \Delta'\tau^2 + g'\tau + \omega$, $\iota = \iota_a\tau^a + \cdots + \iota_0 \in L\{\tau\}$.
Then ι is an isogeny $\phi \longrightarrow \psi$ *iif*

$$\begin{aligned} \Delta' \iota_a^{q^2} - \Delta^{q^a} \iota_a &= 0, \\ \Delta' \iota_{a-1}^{q^2} - \Delta^{q^{a-1}} \iota_{a-1} &= \iota_a g^{q^a} - g' \iota_a^q, \\ \forall k \in [\![2, a]\!], \quad \Delta' \iota_{a-k}^{q^2} - \Delta^{q^{a-k}} \iota_{a-k} &= \iota_{a-k+1} g^{q^{a-k+1}} - g' \iota_{a-k+1}^q + \iota_{a-k+2}(\omega^{q^{a-k+2}} - \omega), \\ \iota_0 g + \iota_1 \omega^q &= \omega \iota_1 + g' \iota_0^q. \end{aligned}$$

Wesolowski, 2022: this is a linear system! In our case, it is solvable in time linear of $[K : \mathbb{F}_q]$.

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Demo!