An explicit CRS-like action with Drinfeld modules

Séminaire de l'équipe LFANT

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Juin 2022

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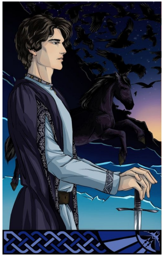

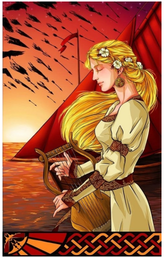

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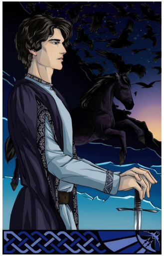

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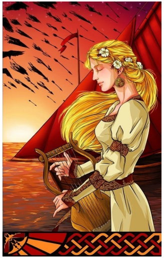

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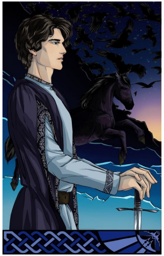

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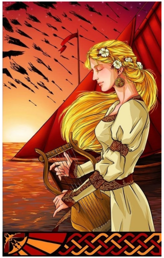

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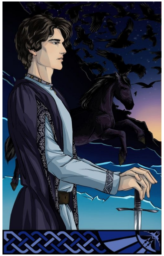

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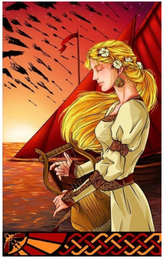

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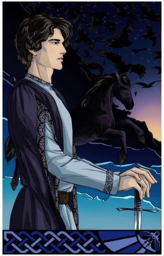

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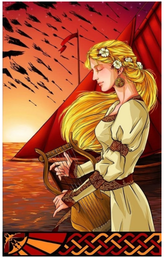

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Theorem (Classical result from class field theory)

Let $E/\mathbb{F}_q$ be some ordinary elliptic curve. Fix $\mathcal{O} = \mathrm{End}_{\mathbb{F}_q}(E)$.
Then, $\mathrm{Cl}(\mathcal{O})$ acts simply transitively on the set of $\overline{\mathbb{F}_q}$-isomorphism classes of elliptic curves defined over $\mathbb{F}_q$ with same endomorphism ring and characteristic polynomial as E.

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Analogies (1/2)

Number fields	Function fields
$\mathbb{Z}$	$\mathbb{F}_q[X]$
Imaginary quadratic number fields	Imaginary hyperelliptic curves
Class group (hard computation)	Jacobian (small characteristic: doable computation with Kedlaya's algorithm)
Elliptic curves	Drinfeld modules

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Analogies (2/2)

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$\mathbb{Z}$-module law on $E(\overline{\mathbb{F}_q})$	$\mathbb{F}_q[X]$-module law on $\overline{\mathbb{F}_q}$
$E[n] \simeq (\mathbb{Z}/n)^2$ if $p \nmid n$ $E[p] \simeq (\mathbb{Z}/p)^{s\in\{0,1\}}$	$\phi[a] \simeq (\mathbb{F}_q[X]/a)^r$ if $\mathfrak{p} \nmid a$ $\phi[\mathfrak{p}] \simeq (\mathbb{F}_q[X]/\mathfrak{p})^{s\in\{0,\ldots,r-1\}}$
Vélu formulae	
j-invariant encoding $\overline{\mathbb{F}_q}$-isomorphism classes	
Characteristic polynomial of the Frobenius endomorphism	
Theory of complex multiplication	
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Main results

Preprint `ia.cr/2022/349`.

Computer algebra:

- Definition of a CRS-like group action for Drinfeld modules. Proof that it is simply transitive.
- Algorithm to compute the action.
- Efficient C++/NTL implementation.
- SageMath implementation of Drinfeld modules (work in progress, `https://trac.sagemath.org/ticket/33713`).

Cryptography:

- Reduction of the inverse problem to the isogeny-finding problem.
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Let:

- ϕ: *potential* Drinfeld module;
- $a, b \in \mathbb{F}_q[X]$, $x, y \in \overline{\mathbb{F}_q}$, $\lambda \in \mathbb{F}_q$.

Act on $\overline{\mathbb{F}_q}$ (instead of $E(\overline{\mathbb{F}_q})$):

Goal 1: $a \cdot (x + y) = a \cdot x + a \cdot y$;
Goal 2: $\lambda \cdot x = \lambda x$;
(1) + (2): $\phi(a) : (x \mapsto a \cdot x)$ is $\mathbb{F}_q$-linear ($\phi(a) \in \mathrm{End}_{\mathbb{F}_q}(\overline{\mathbb{F}_q})$).

Goal 3: $a \cdot (b \cdot x) = (ab) \cdot x$;
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Linear endomorphisms of $\overline{\mathbb{F}_q}$ (1/3)

Any $f \in \mathrm{End}_{\mathbb{F}_q}(\overline{\mathbb{F}_q})$ is

$$f : x \mapsto l_n x^{q^n} + \cdots + l_2 x^q + l_1 x, \quad l_i \in \overline{\mathbb{F}_q}.$$

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Linear endomorphisms of $\overline{\mathbb{F}_q}$ (1/3)

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Let $L/\mathbb{F}_q$ be finite. Denote

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Definition (Ore, 1933)

The ring $(L\{\tau\}, +, \circ)$ is called the *ring of Ore polynomials in τ with coefficients in L*.

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$L\{\tau\}$ is left-euclidean: $\forall P_1, P_2 \in L\{\tau\}, \deg_\tau(P_1) \geqslant \deg_\tau(P_2), \exists Q, R \in L\{\tau\}$ s.t.:

$$\begin{cases} P_1 = QP_2 + R, \\ \deg_\tau(R) < \deg_\tau(P_2). \end{cases}$$

We can compute RGCD in $L\{\tau\}$.

$$\langle \{P_i(\tau)\} \rangle = \mathrm{rgcd}(\{P_i(\tau)\}) L\{\tau\}.$$

SageMath implementation (Xavier Caruso).

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Definition of a finite Drinfeld $\mathbb{F}_q[X]$-module

Fix $\omega \in Ł^\times$.

Definition (Drinfeld, 1974)

A *finite Drinfeld $\mathbb{F}_q[X]$-module defined over L* is an $\mathbb{F}_q$-algebra morphism

$$\phi : \mathbb{F}_q[X] \to L\{\tau\}$$

s.t. $\mathrm{Im}(\phi) \not\subset L$, $\mathrm{ConstCoeff}(\phi(X)) = \omega$.

Theorem (Drinfeld, 1974)

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Fix $\phi : \mathbb{F}_q[X] \to L\{\tau\}$ a finite Drinfeld $\mathbb{F}_q[X]$-module.
ϕ uniquely determined by

$$\phi(X) = \phi_n \tau^n + \cdots + \phi_1 \tau + \omega, \quad \phi_n \neq 0.$$

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The *rank of ϕ* is n.

Rank 2 finite Drinfeld modules are closest to elliptic curves over finite fields.

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Morphisms and isogenies

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An *isogeny* is a nonzero morphism.

Endomorphisms always contain $\mathbb{F}_q[X]$ and $\tau_L = x \mapsto x^{\#L}$:

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Characteristic polynomial (1/2)

Assume $\mathrm{rank}(\phi) = 2$.

There exists

$$\chi_\phi(X)(T) = T^2 - A(X)T + B(X) \in \mathbb{F}_q[X][T]$$

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and $\deg_X(A) \leqslant d/2$, $\deg_X(B) = d$ (Hasse bounds).

Definition

χ_ϕ is the *characteristic polynomial of the Frobenius endomorphism of ϕ*.

ϕ is *supersingular iif* $\mathfrak{p} = \mathrm{MinPol}_{\mathbb{F}_q}(\omega)$ divides A.

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Main result

Theorem (Classical result from class field theory)

Let $E/\mathbb{F}_q$ be some ordinary elliptic curve. Fix $\mathcal{O} = \mathrm{End}_{\mathbb{F}_q}(E)$.
Then, $\mathrm{Cl}(\mathcal{O})$ acts simply transitively on the set of $\overline{L}$-isomorphism classes of elliptic curves defined over $\mathbb{F}_q$ with same endomorphism ring and characteristic polynomial as E.

Theorem (L., Spaenlehauer, 2022)

Assume $[L : \mathbb{F}_q]$ is odd and $\geqslant 5$. Let ϕ be some ordinary rank two finite Drinfeld module. Fix $\mathcal{O} = \mathrm{End}_L(\phi)$.
Assume χ_ϕ defines an imaginary hyperelliptic curve $\mathcal{H}$.
Then, $\mathrm{Cl}(\mathcal{O}) \simeq \mathrm{Pic}^0(\mathcal{H})$ and $\mathrm{Cl}(\mathcal{O})$ acts simply transitively on the set of $\overline{L}$-isomorphism classes of rank 1 Drinfeld modules $\mathcal{O} \to L\{\tau\}$.

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Definition of the action

Let $\mathfrak{a} \in \mathrm{Id}(\mathcal{O})$, let ϕ' be a representative. Let

$$V_{\mathfrak{a}} = \bigcap_{f \in \mathfrak{a}} \mathrm{Ker}(f).$$

$V_{\mathfrak{a}}$ is the kernel of some isogeny $\iota_{\mathfrak{a}}$ with domain ϕ'. We associate

$$\mathfrak{a} \star \phi' := \text{codomain of } \iota_{\mathfrak{a}}.$$

Definition

This map can be extended to the class group and to set of isomorphism classes. This defines our action.

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Isomorphism $\mathrm{Cl}(\mathcal{O}) \simeq \mathrm{Pic}^0(\mathcal{H})$

$\mathrm{End}(\phi) = \mathcal{O}$ always contains $\mathbb{F}_q[X]$ and τ_L.

In our case, that's it:

$$\mathcal{O} \simeq \mathbb{F}_q[X][Y]/\chi_\phi \simeq \text{ring of functions on } \mathcal{H} \text{ regular outside } \infty,$$

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Mumford coordinates for $\mathrm{Pic}^0(\mathcal{H})$

Representation with Mumford coordinates:

$$\mathrm{Pic}^0(\mathcal{H}) \longleftrightarrow \mathrm{Cl}(\mathbb{F}_q[X][Y]/\chi_\phi)$$
$$(u,v) \longleftrightarrow \text{Class of } \langle u(\overline{X}), \overline{Y} - v(\overline{X})\rangle$$

with $u, v \in \mathbb{F}_q[X]$ and $u \neq 0$ is monic, $\deg(v) < \deg(u) \leqslant (d-1)/2$, $u \mid \chi(X, v(X))$ and $d = [L : \mathbb{F}_q]$ (Hasse-Weil bounds)

Explicit computation

$$V_{\mathfrak{a}} = \mathrm{Ker}(\iota_{\mathfrak{a}}) = \bigcap_{f \in \mathfrak{a}} \mathrm{Ker}(f) = \bigcap_{\overline{f} \in \langle u(\overline{X}), \overline{Y} - v(\overline{X}) \rangle} \mathrm{Ker}(f(\phi(X), \tau_L)).$$

Therefore

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Algorithm

Input: — A j-invariant $j \in L$.
— Mumford coordinates $(u, v) \in \mathbb{F}_q[X]^2$.

Output: A j-invariant.

$\widetilde{u} \leftarrow u(j^{-1}\tau^2 + \tau + \omega) \in L\{\tau\}$;
$\widetilde{v} \leftarrow v(j^{-1}\tau^2 + \tau + \omega) \in L\{\tau\}$;
$\iota \leftarrow \mathrm{rgcd}(\widetilde{u}, \tau^{[L:\mathbb{F}_q]} - \widetilde{v})$;
$\widehat{g} \leftarrow \iota_0^{-q}(\iota_0 + \iota_1(\omega^q - \omega))$;
$\widehat{\Delta} \leftarrow j^{-q^{\deg_\tau(\iota)}}$;
Return $\widehat{g}^{q+1}/\widehat{\Delta}$.

C++ / NTL implementation of the action: computation in ∼200 ms for $\mathbb{F}_q = \mathbb{F}_2$ and $[L : \mathbb{F}_q] = 521$. The hyperelliptic curve has genus $\frac{521-1}{2} = 260$, $\mathrm{Pic}^0(\mathcal{H})$ has order

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Inverse problem

Theorem (L., Spaenlehauer, 2022)

The problems of inverting the action and the problems of finding finite Drinfeld module polynomially reduce to one another.

Write $\phi(X) = \Delta\tau^2 + g\tau + \omega$, $\psi(X) = \Delta'\tau^2 + g'\tau + \omega$, $\iota = \iota_a\tau^a + \cdots + \iota_0 \in L\{\tau\}$.
Then ι is an isogeny $\phi \to \psi$ *iif*

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\forall k \in [\![2,a]\!],\quad &\Delta'\iota_{a-k}^{q^2} - \Delta^{q^{a-k}}\iota_{a-k} = \iota_{a-k+1}g^{q^{a-k+1}} - g'\iota_{a-k+1}^q + \iota_{a-k+2}(\omega^{q^{a-k+2}} - \omega),\\
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Attacks on this problem

- Previous work (Joux, Narayanan, 2019; Caranay, Greenberg, Scheidler, 2020): the system is solved recursively.
- Wesolowski (2022): this is an $\mathbb{F}_q$-linear system of equations. We can find an $\mathbb{F}_q$-basis by writing each coefficient in an $\mathbb{F}_q$-basis of L.

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