Function field analogue of the CRS key exchange

Journées C2 2022

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INRIA Nancy-Grand Est

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This unification is made with the notion of *hard homogeneous space (HHS)*.

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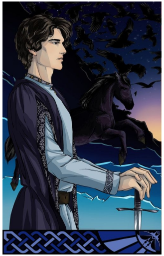

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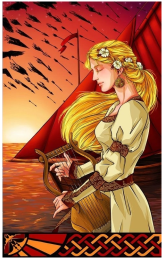

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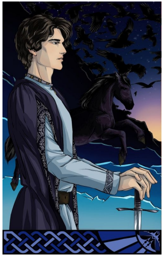

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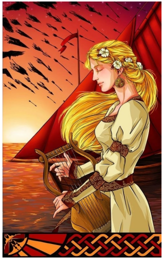

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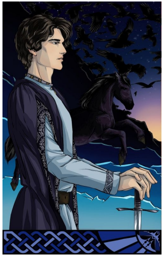

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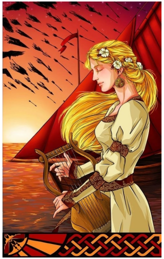
⟵ $b \cdot x$

⇠ Both calculate $ab \cdot x$ (secret key) ⇢



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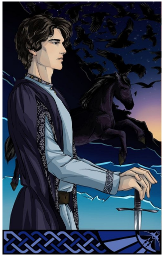

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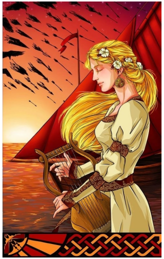

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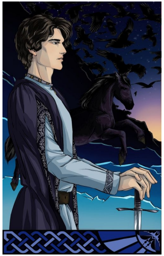

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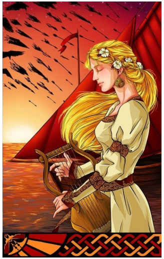

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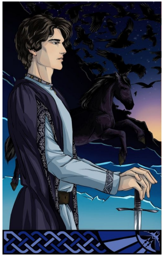

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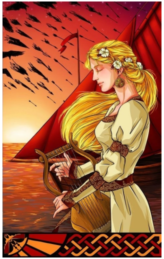

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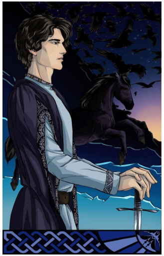

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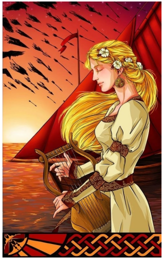

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Quantum attack in $\exp(c\sqrt{\log(\#G)})$ for some $c > 0$ (Kuperberg, 2005; Bonnetain, Schrottenloher, 2020).

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Furthermore:

- SageMath library for finite Drinfeld modules (work in progress).

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Theorem

There is an explicit and computable group action of $\mathrm{Pic}^0(\mathcal{H}) \simeq \mathrm{Cl}(\mathbf{A}_{\mathcal{H}})$ *to the set of* $\overline{\mathbb{F}_q}$*-isomorphism classes of rank* 1 $\mathbf{A}_{\mathcal{H}}$*-Drinfeld modules defined over* L*.*

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- SageMath implementation by X. Caruso.

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Input: — A j-invariant $j \in L$.
— Mumford coordinates $(u, v) \in \mathbb{F}_q[X]^2$.

Output: A j-invariant.

```
// ω is a global constant
```

$\widetilde{u} \leftarrow u(j^{-1}\tau^2 + \tau + \omega) \in L\{\tau\}$;

$\widetilde{v} \leftarrow v(j^{-1}\tau^2 + \tau + \omega) \in L\{\tau\}$;

$\iota \leftarrow \mathrm{rgcd}(\widetilde{u}, \tau^{[L:\mathbb{F}_q]} - \widetilde{v})$;

$\widehat{g} \leftarrow \iota_0^{-q}(\iota_0 + \iota_1(\omega^q - \omega))$;

$\widehat{\Delta} \leftarrow j^{-q^{\deg_\tau(\iota)}}$;

Return $\widehat{g}^{q+1}/\widehat{\Delta}$.

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But two weeks ago... Wesolowski, solved this problem in polynomial time, reducing the isogeny-search problem to a linear algebra problem.

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