Digital Signal Processing: Spectral Analysis

CPSC 501: Advanced Programming Techniques
Fall 2020

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Fourier Theorem

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Fourier Theorem

- **Fourier Theorem:**
  - Any continuous, periodic waveform can be expressed as the sum of a series of sine and cosine terms, each having specific amplitude and phase coefficients
  - Any physical function that varies periodically with time with a frequency f can be expressed as a superposition of sinusoidal components of frequencies: $f, 2f, 3f, 4f, \ldots$"

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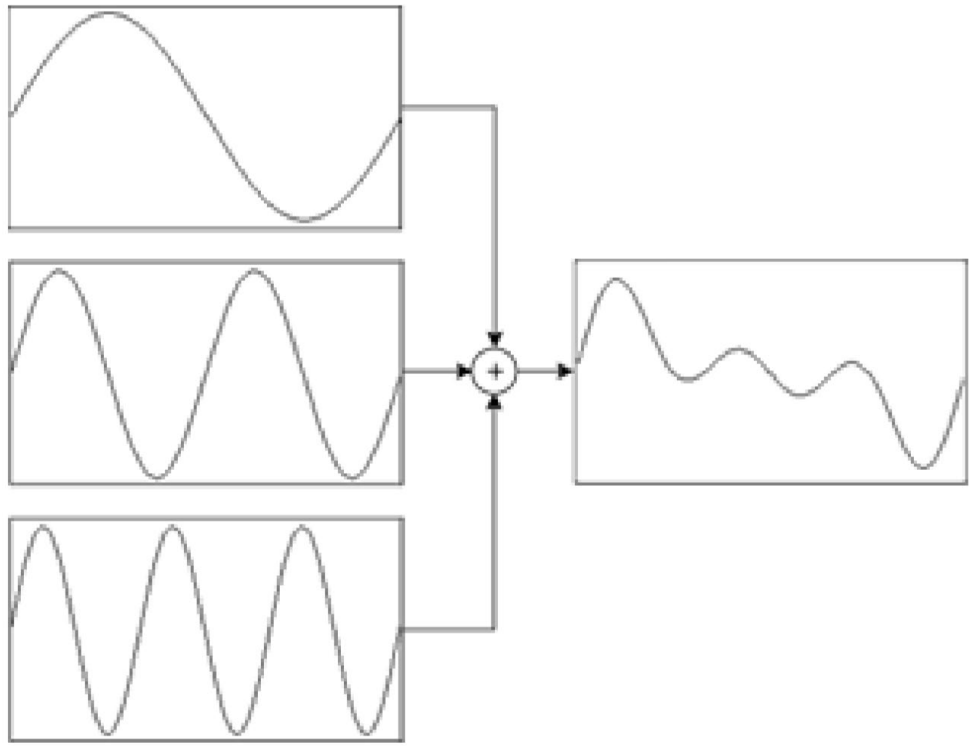
Additive synthesis

- Pure tones can be added together to form a complex tone (additive synthesis):



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Fourier decomposition

- Fourier analysis decomposes a complex signal into its component parts
  - i.e. its spectrum
  - The **Fourier transform** calculates the spectrum of a continuous signal
    - Transforms from the **time domain** to **the frequency domain**

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Discrete Fourier Transform

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Discrete Fourier Transform

- The discrete Fourier transform (DFT) calculates the spectrum of a digital signal
  - Definition:

$$DFT[x(n)] = X(k)$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j\omega kn} \qquad 0 \leq k \leq N-1$$

    - N is the number of samples per period of the waveform
    - k is the harmonic number
    - and ω = 2π/N

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What frequencies are present

- The discrete Fourier transform (DFT) calculates the spectrum of a digital signal
- Spectrum -> Which frequencies are present
- Harmonic Numbers -> we go up multiples of frequencies (this gives us regular components to break signal down into)

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Breakdown

- Since $e^{jx} = \cos(x) + j\sin(x)$, can be expressed as:

$$\sum_{n=0}^{N-1} x(n)\cos(\omega nk) - j\sum_{n=0}^{N-1} x(n)\,sin(\omega nk) \qquad 0 \le k \le N-1$$

Real part a_k Imaginary part b_k

- We must calculate the real part a_k and the imaginary part b_k separately

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In code

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Get initial summation values

- C implementation:

```
#define PI     3.141592653589793
#define TWO_PI (2.0 * PI)
void dft(double x[], int N, double a[], double b[]){
    int n, k;
    double omega = TWO_PI / (double)N;
    for (k = 0; k < N; k++) {
        a[k] = b[k] = 0.0;
        for (n = 0; n < N; n++) {
            a[k] += (x[n] * cos(omega * n * k));
            b[k] -= (x[n] * sin(omega * n * k));
} } }
```


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Process result

- The results must be further processed :
  - a_k and b_k must be scaled by N:

```
for (k = 0; k < N; k++) {
    a[k] /= (double)N;
    b[k] /= (double)N;
}
```

  - The magnitude or amplitude |X(k)| is given by:

$$|X(k)| = \sqrt{a_k^2 + b_k^2}$$

```
for (k = 0; k < N; k++)
    amplitude[k] = sqrt( (a[k] * a[k]) + (b[k] * b[k]) )
```


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What we get?

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Harmonic numbers

- The DFT gives amplitudes for both positive and negative frequencies (harmonics)
  - If N = 8:

k	Harmonic
0	0 (DC)
1	1
2	2
3	3
4	4
5	-3
6	-2
7	-1

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Final amplitudes

- Add these together to get the amplitude of a harmonic:

```
x[0] = amplitude[0];
x[N/2] = amplitude[N/2];
for (k = 1, j = N-1; k < N/2; k++, j--)
    x[k] = amplitude[k] + amplitude[j];
```


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Performance?

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Performance

- The DFT is $O(N^2)$
  - Not practical for large data sets
  - Can be calculated more efficiently with the fast Fourier transform (FFT)
    - Is O(N log N)

- There are many variants of the FFT
  - Most work by reordering the data, and then recursively subdividing it in half
    - Thus data size must be a power of 2

- Most FFTs work with complex numbers
  - The signal is the real part
  - The imaginary part is set to 0
  - This data is packed into an array of size*2:

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FFT

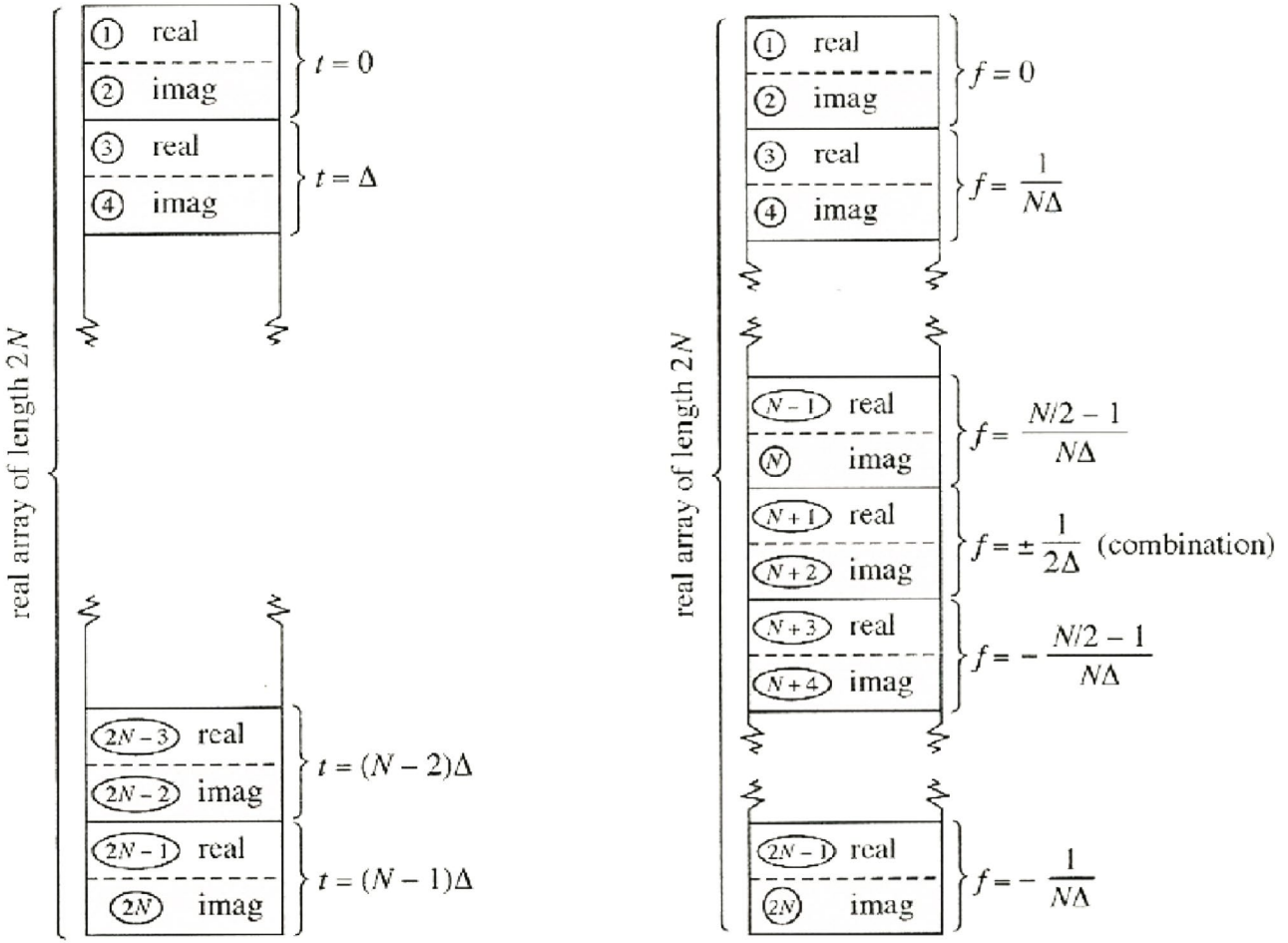

Figure 12.2.2. Input and output arrays for FFT. (a) The input array contains N (a power of 2) complex time samples in a real array of length $2N$, with real and imaginary parts alternating. (b) The output array contains the complex Fourier spectrum at N values of frequency. Real and imaginary parts again alternate. The array starts with zero frequency, works up to the most positive frequency (which is ambiguous with the most negative frequency). Negative frequencies follow, from the second-most negative up to the frequency just below zero.

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Spectral Analysis

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Spectral Analysis

- Like the DFT, the output of the FFT must be further processed to give meaningful results
- The DFT calculates the spectrum for a single cycle of a waveform
  - Result can be displayed as a bar graph
  - Shows the relative amplitudes of the harmonics

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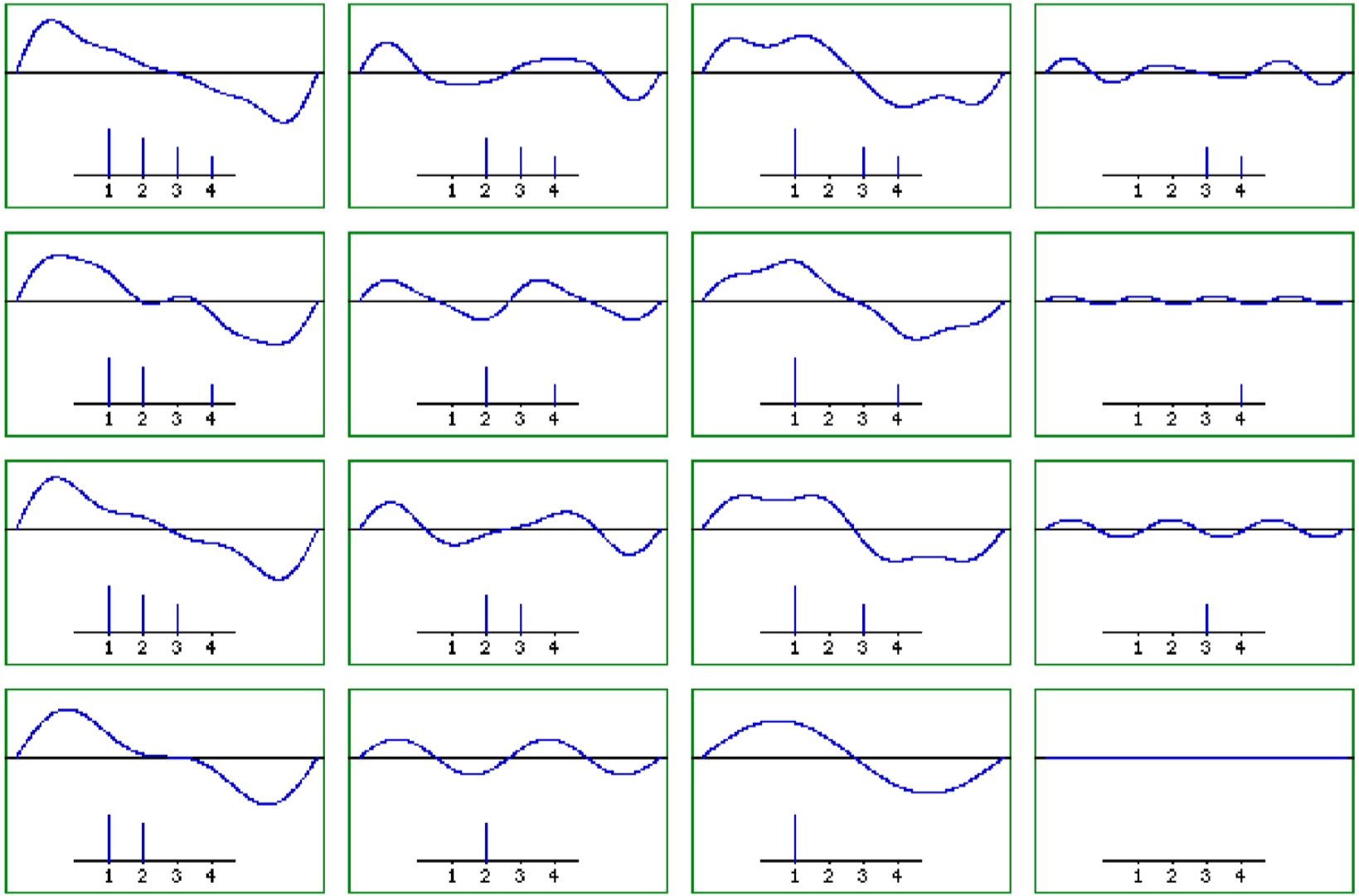
Signals as frequency bar graphs



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DFT Window

- The DFT can be applied to an arbitrary “window” of a longer waveform
  - Is a “snapshot” of the spectrum at a particular time
  - Window is typically 512 or 1024 samples long
  - Instead of harmonics, the output represents how much energy is present in particular “bins”
    - When graphed, the amplitudes are joined together to form a continuous line
    - E.g.

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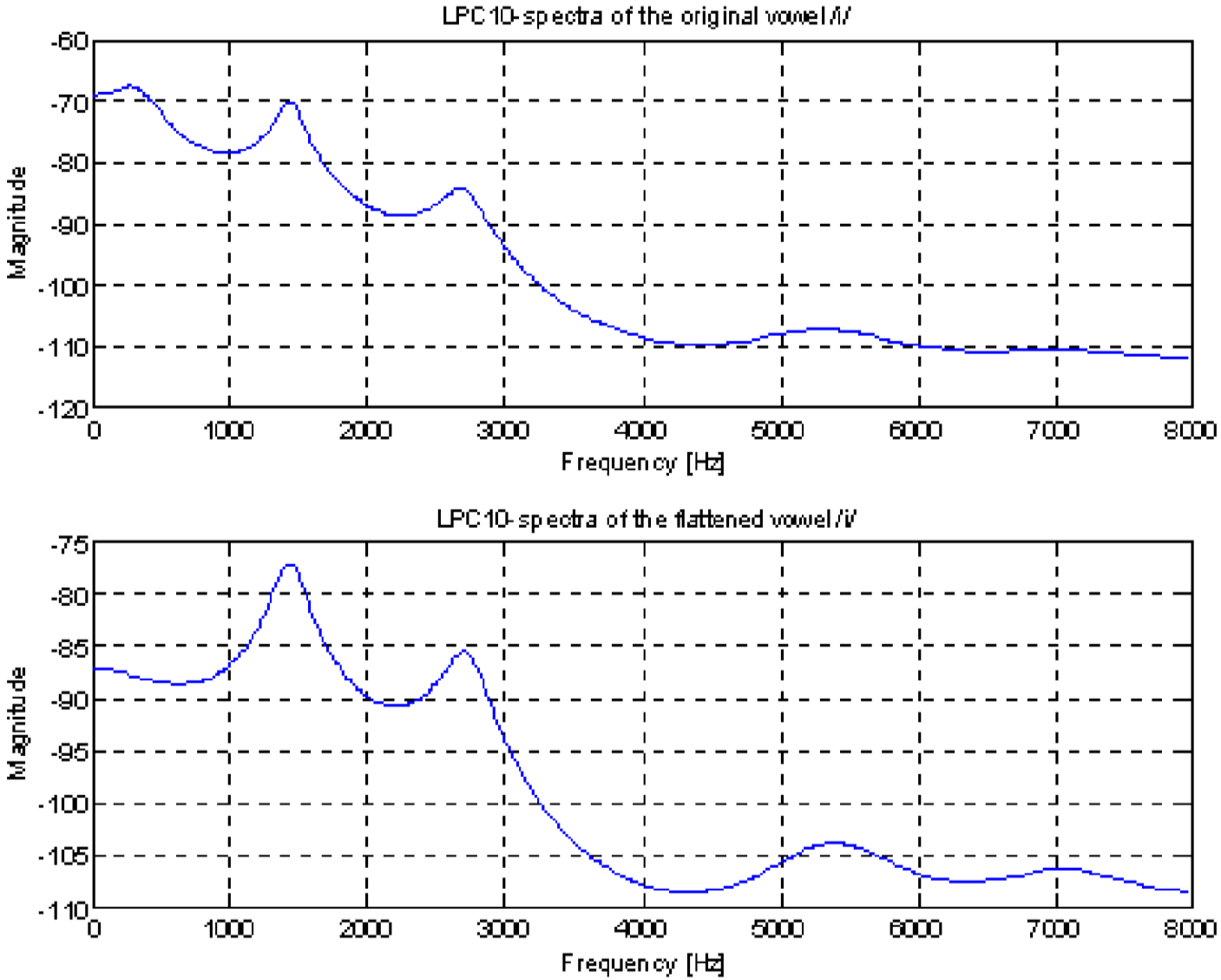
Example frequency outputs



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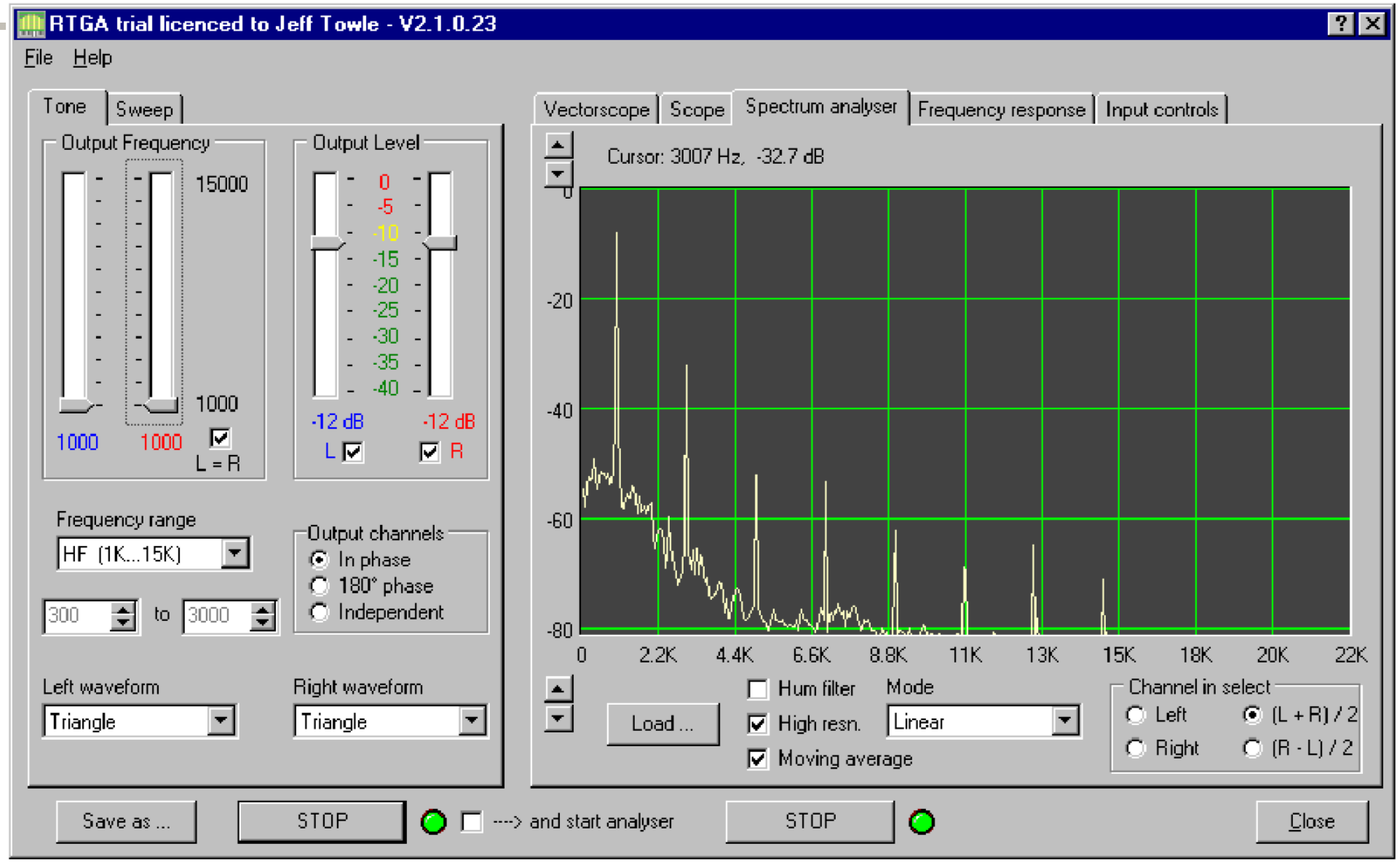
Within an application



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Spectral Analysis

- The spectrum can be plotted as it changes over time
  - The analysis window slides along the time axis
    - Can be end to end
    - Or overlapping
  - Can be graphed as a 3D “waterfall”
    - E.g.

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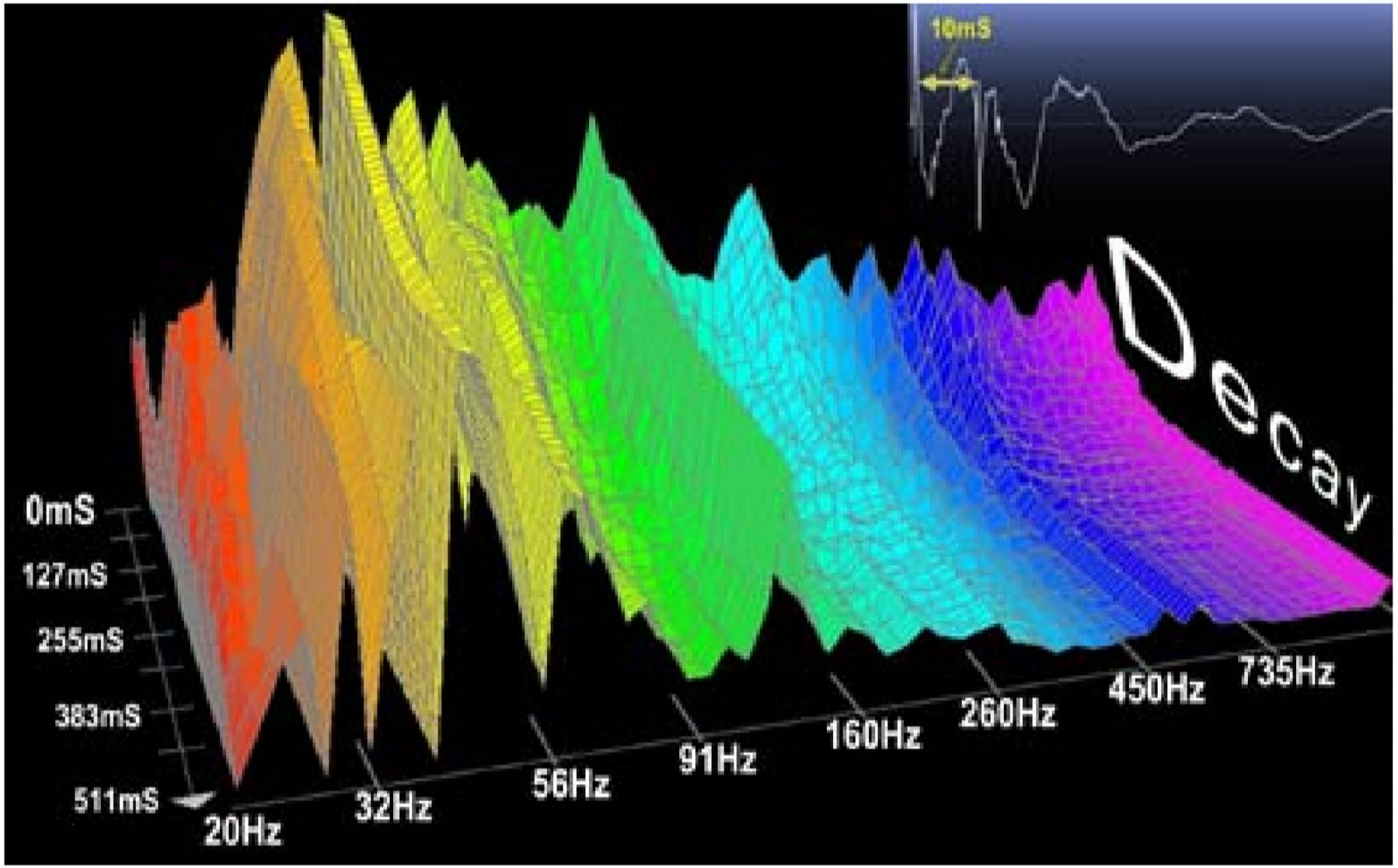
Spectral Analysis



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Spectral Analysis

- Can also be displayed as a spectrogram
  - X axis: time
  - Y axis: frequency
  - Darkness of pixel represents amplitude
    - White: 0 amplitude
    - Black: full amplitude
  - E.g.

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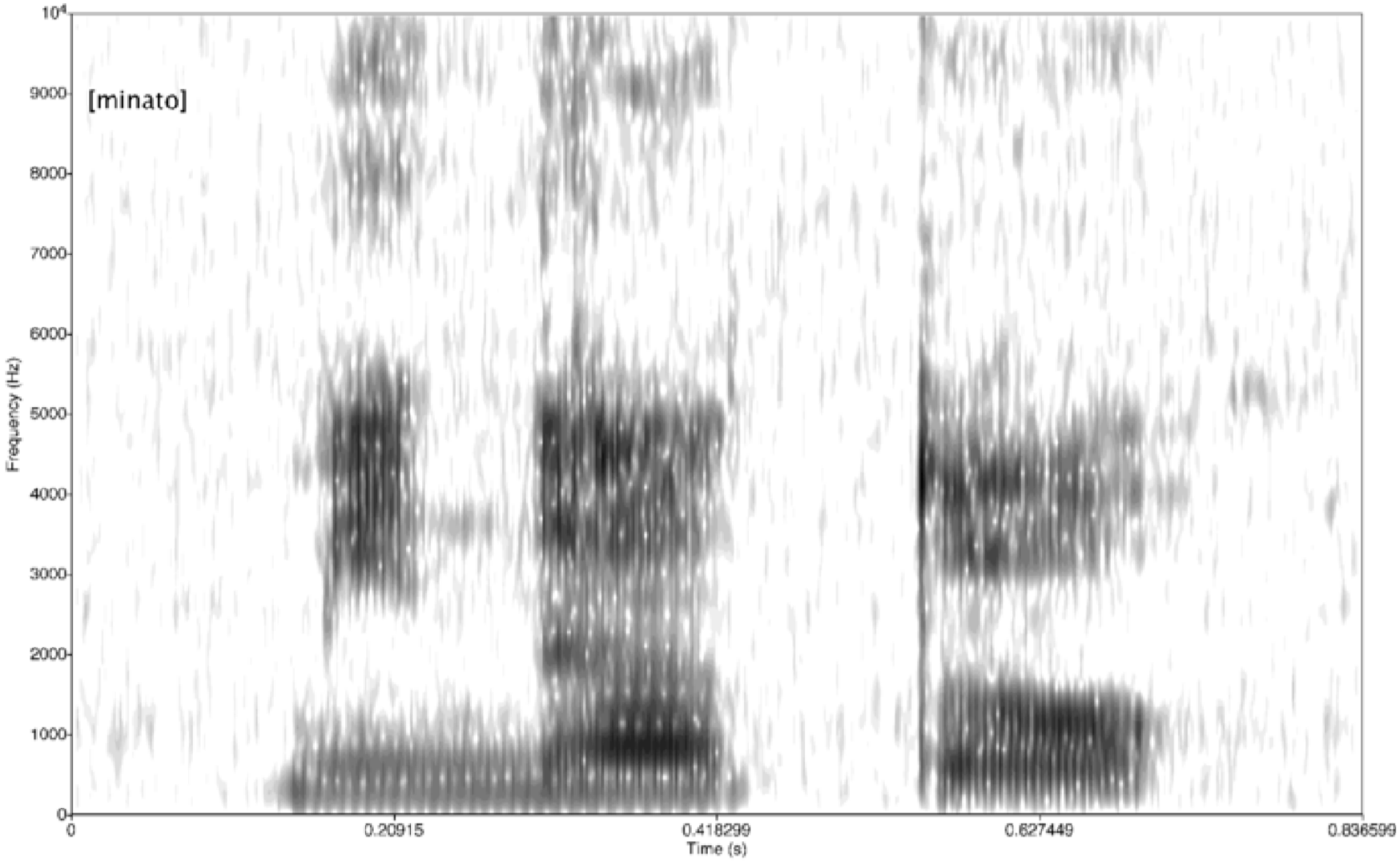
Spectral Analysis



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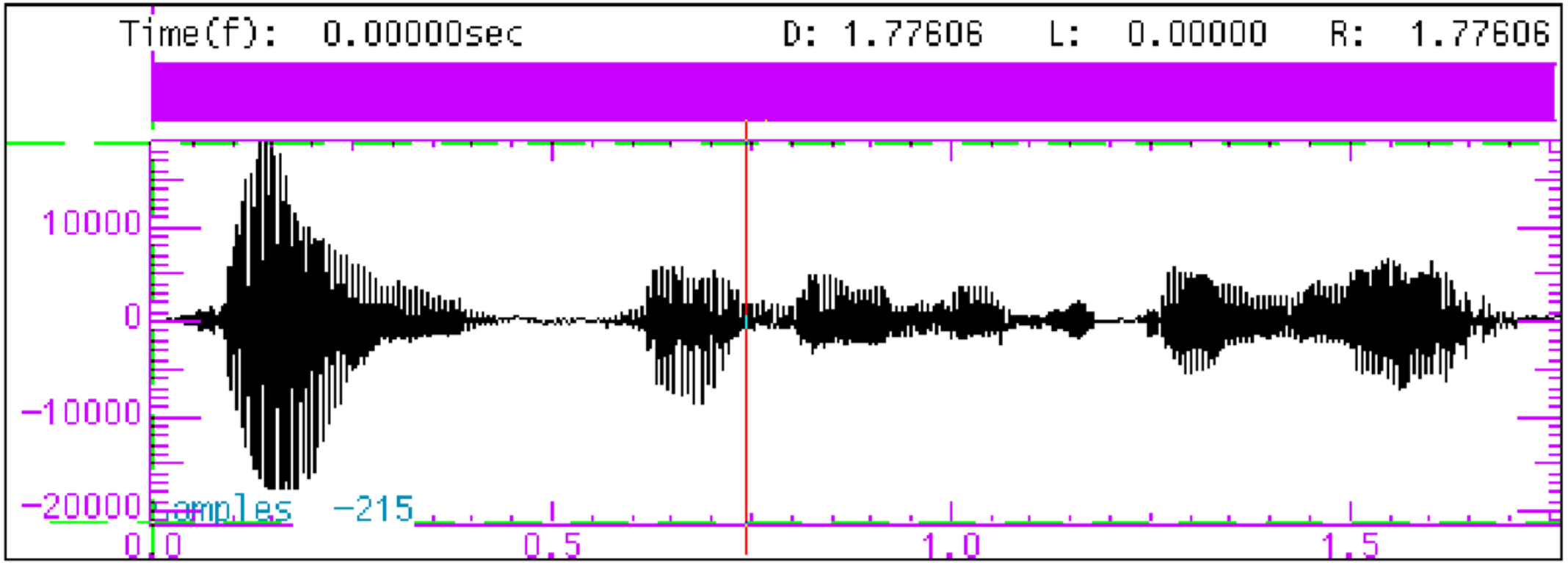
Spectral Analysis



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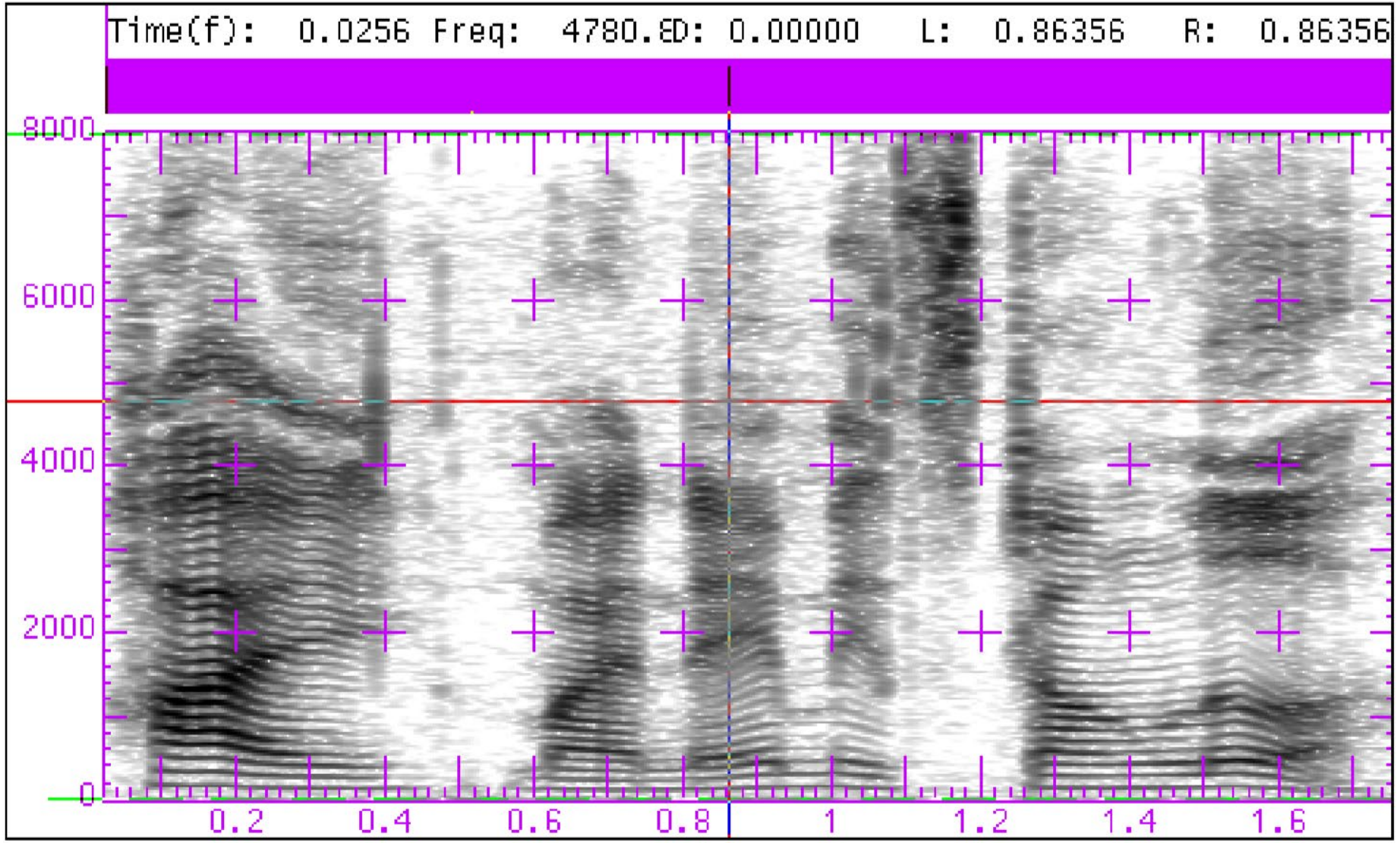
Spectral Analysis



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Onward to … convolution.

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