Digital Signal Processing: Convolution

CPSC 501: Advanced Programming Techniques
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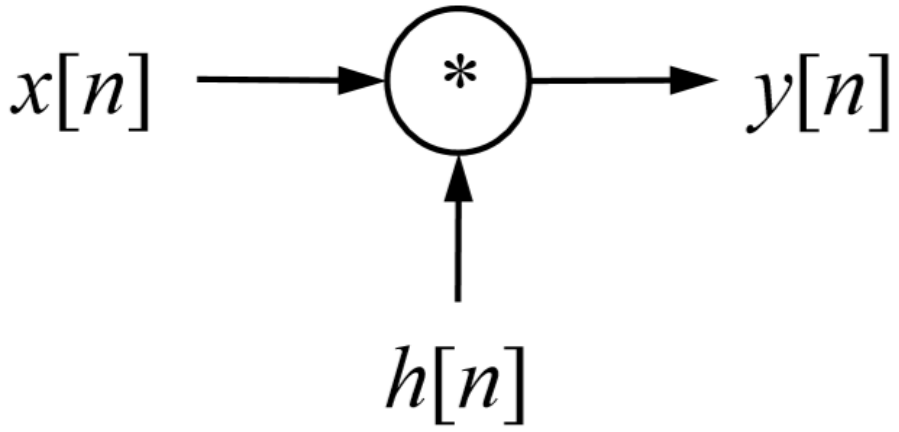
Convolution

Time-Domain Convolution

- Convolution is mathematical operation that takes 2 input signals, x[n] and h[n], to produce a third signal, y[n]
  - Represented mathematically as:

  x[n] * h[n] = y[n]
  - Note: * means convolution, not multiplication
  - Represented graphically:



Time-Domain Convolution

- Input Side Algorithm (array implementation):

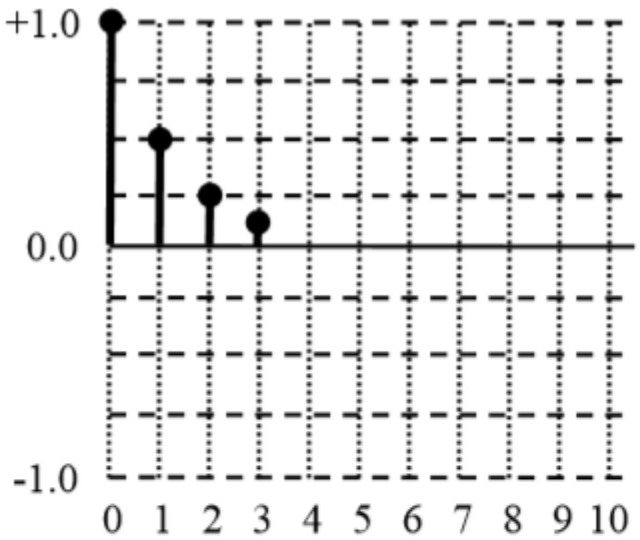
```
void convolve(float x[], int N, float h[], int M, float y[], int P) {
        int n, m;
        /*  Outer loop:  process each input value x[n] in turn  */
        for (n = 0; n < N; n++) {
                /*  Inner loop:  process x[n] with each sample of h[n]  */
                for (m = 0; m < M; m++)
                        y[n+m] += x[n] * h[m];
        }
}
```

Time-Domain Convolution

- Note:
  - N, M, and P are the number of elements in each array
  - P must equal N + M – 1
  - The signals x[n], h[n], and y[n] are f.p. numbers scaled to the range:
    - -1.0 to +1.0
  - You may need to convert to/from signed integers
  - This algorithm can be adapted to deal with samples stored in audio files

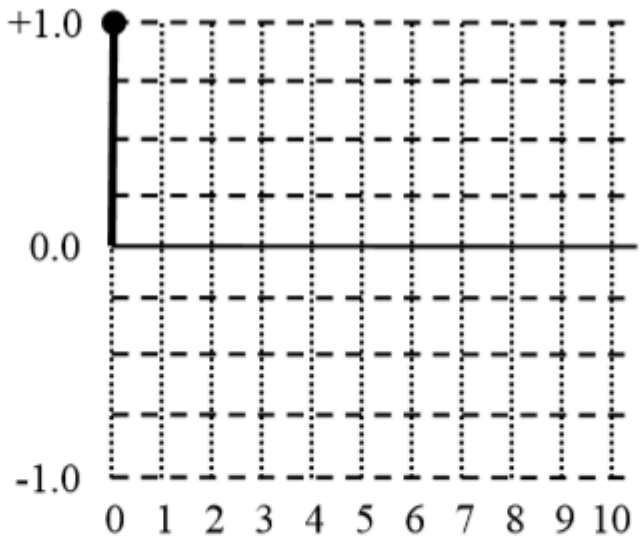
Examples

Example convolutions:



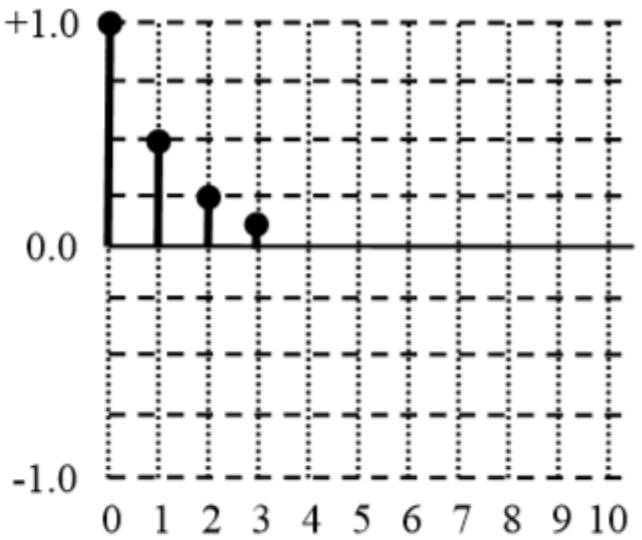
$x[n]$

$*$



$h[n]$

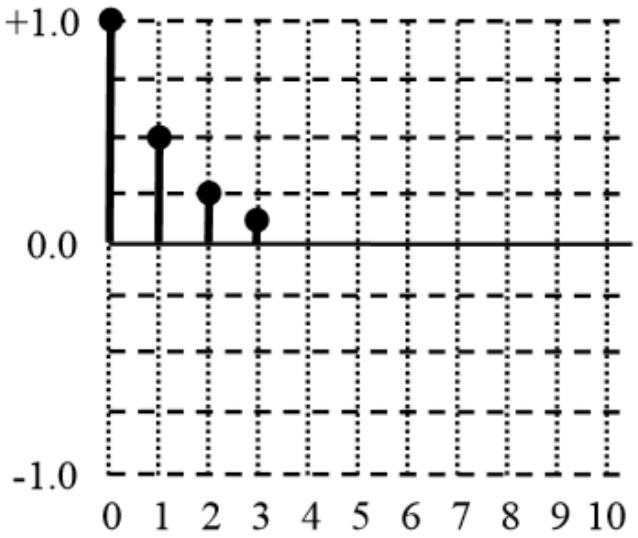
$=$



$y[n]$

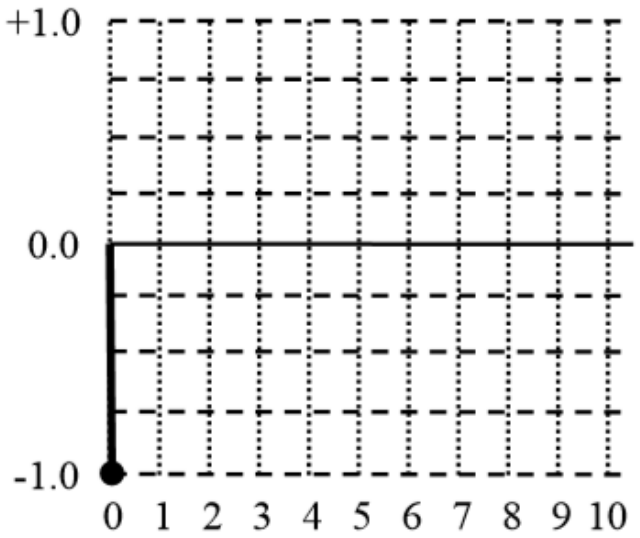
"identity"
impulse response

Example convolutions:



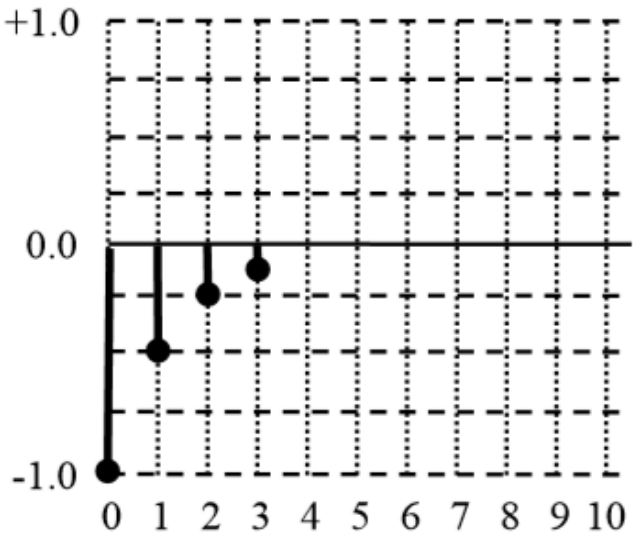
$x[n]$

$*$



$h[n]$

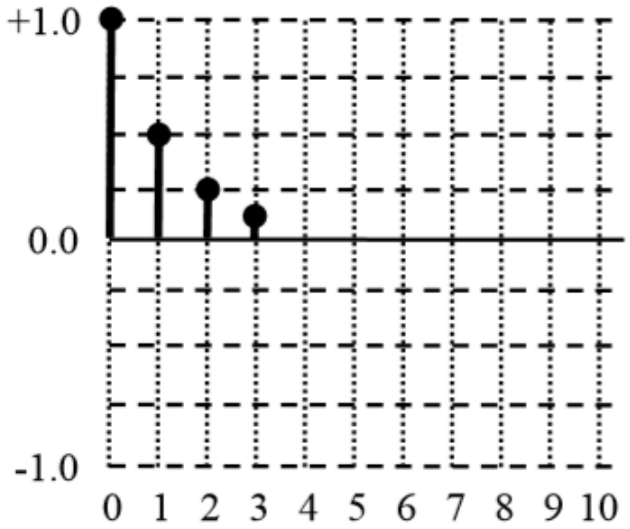
$=$



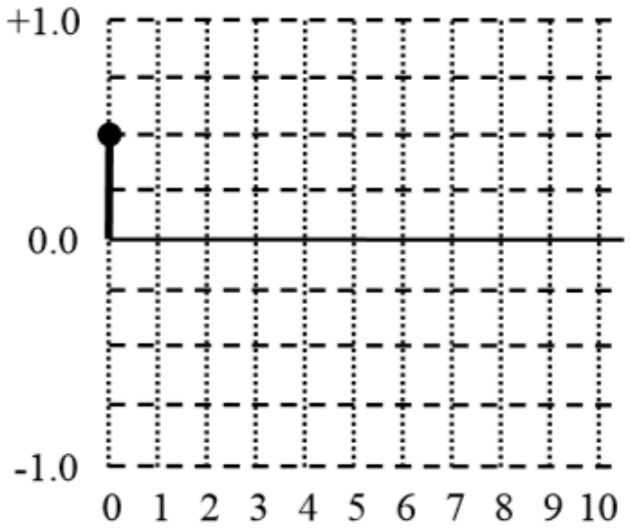
$y[n]$

"inverse" IR

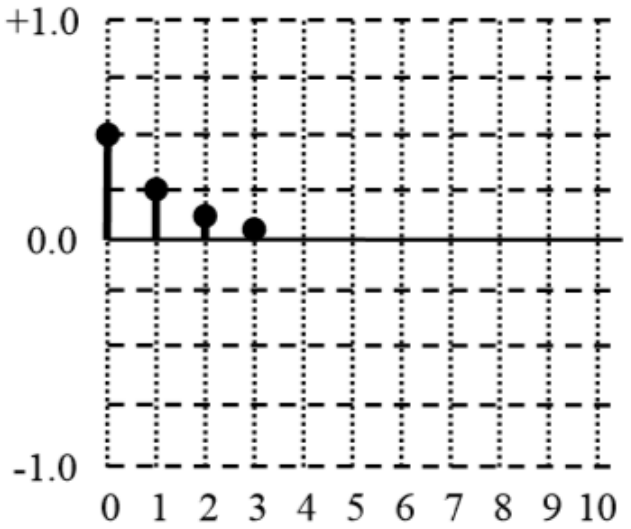
Example convolutions:



$x[n]$ $*$



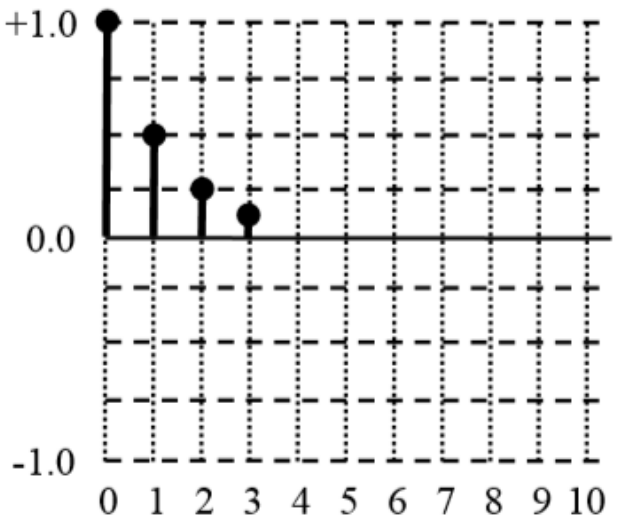
$h[n]$ $=$



$y[n]$

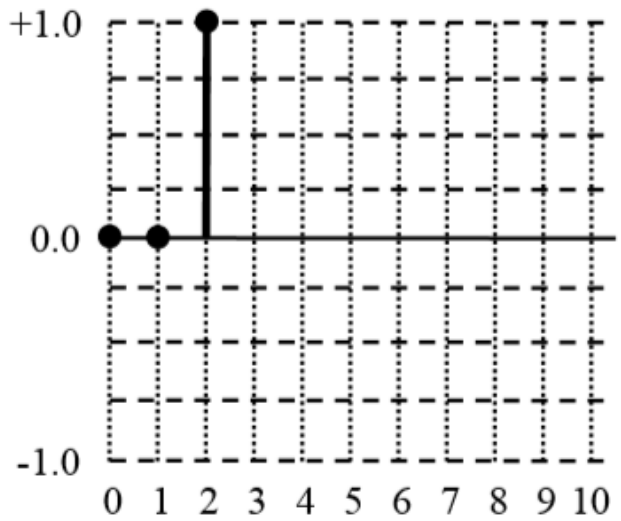
"scaling" IR

Example convolutions:



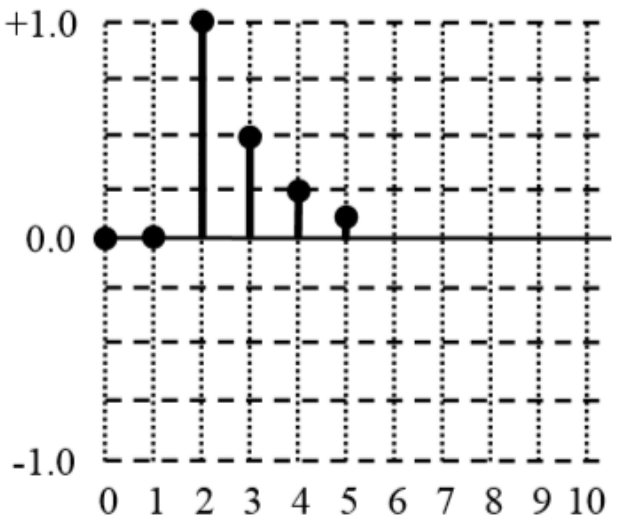
$x[n]$

$*$



$h[n]$

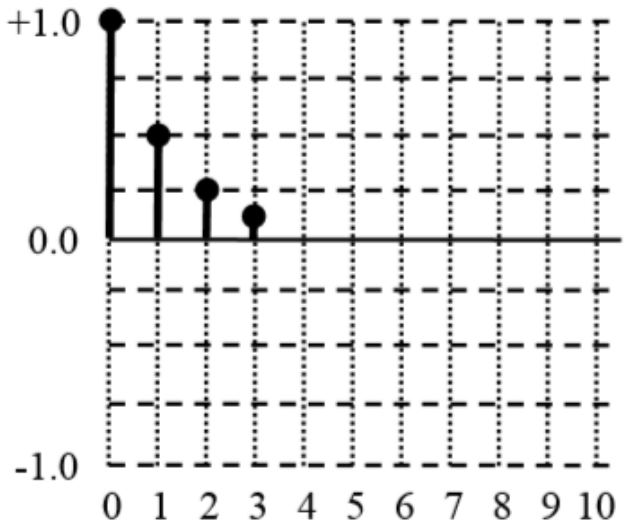
$=$



$y[n]$

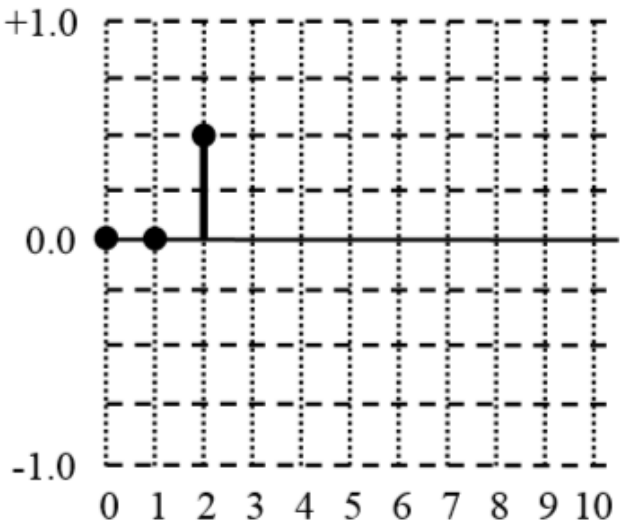
"delay" IR

Example convolutions:



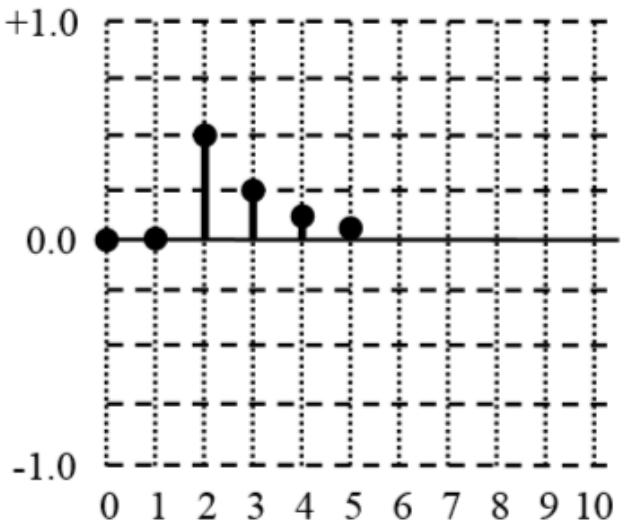
$x[n]$

$*$



$h[n]$

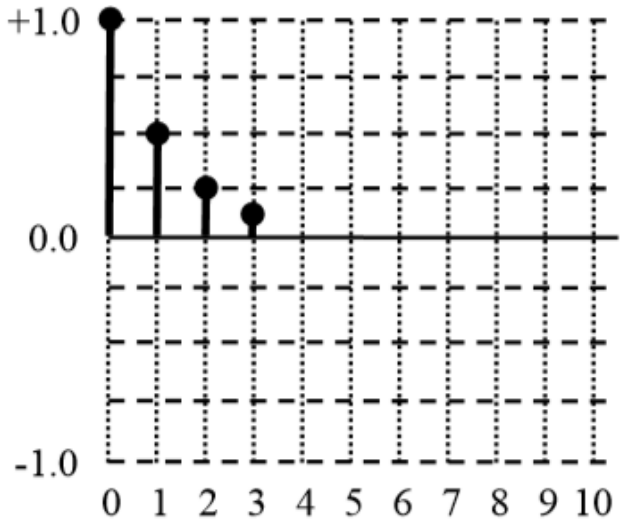
$=$



$y[n]$

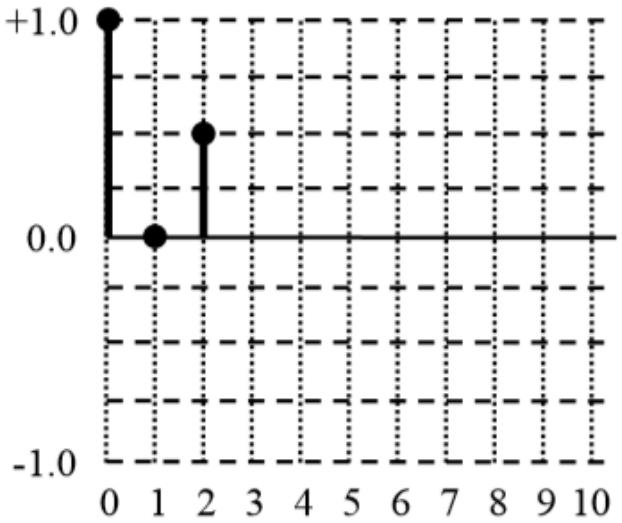
"delay & scaling" IR

Example convolutions:



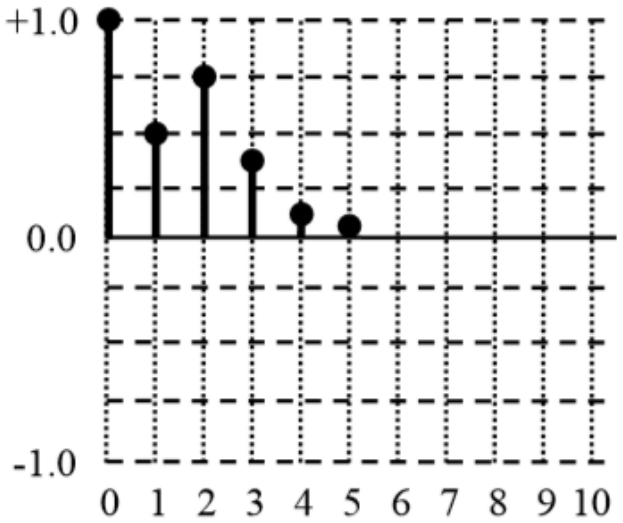
$x[n]$

$*$



$h[n]$

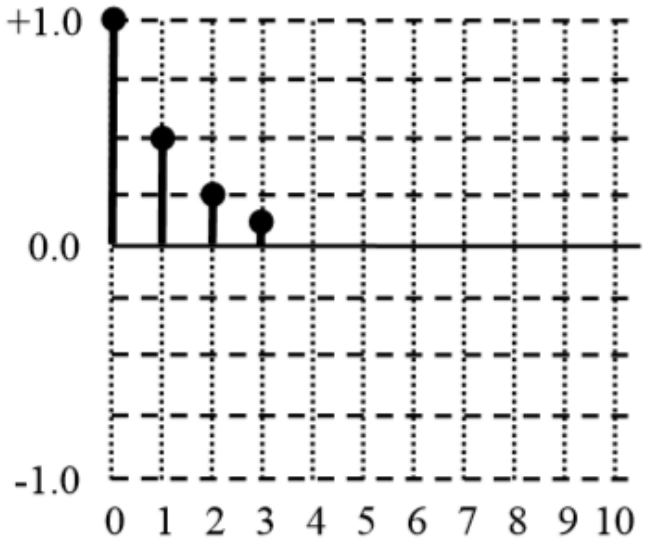
$=$



$y[n]$

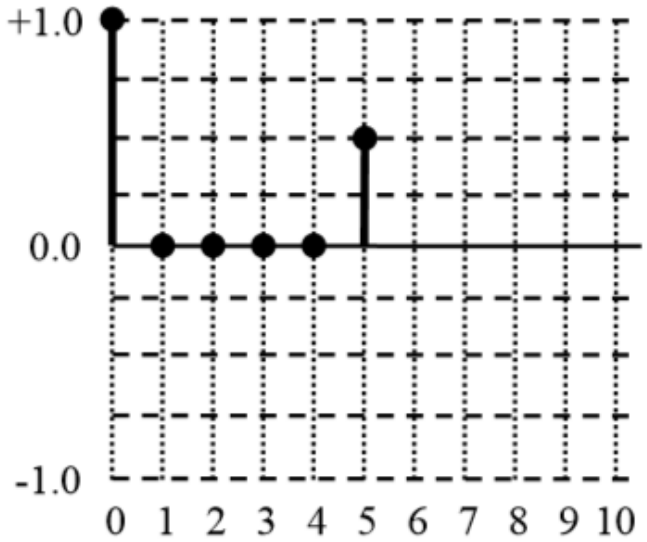
"overlapping echo" IR

Example convolutions:



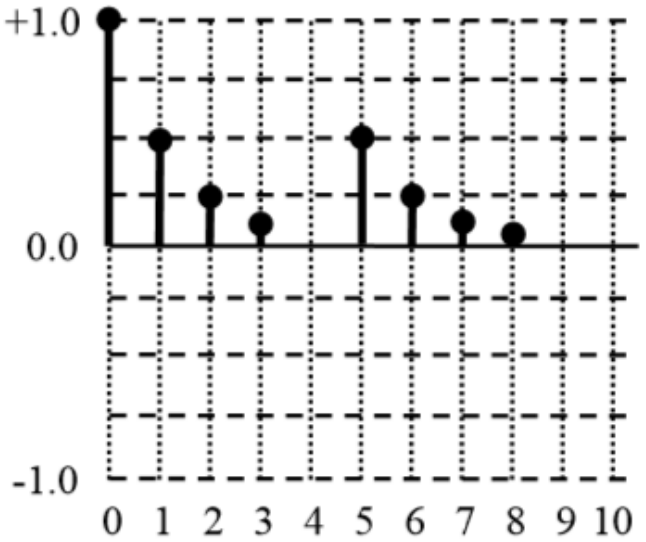
$x[n]$

$*$



$h[n]$

$=$

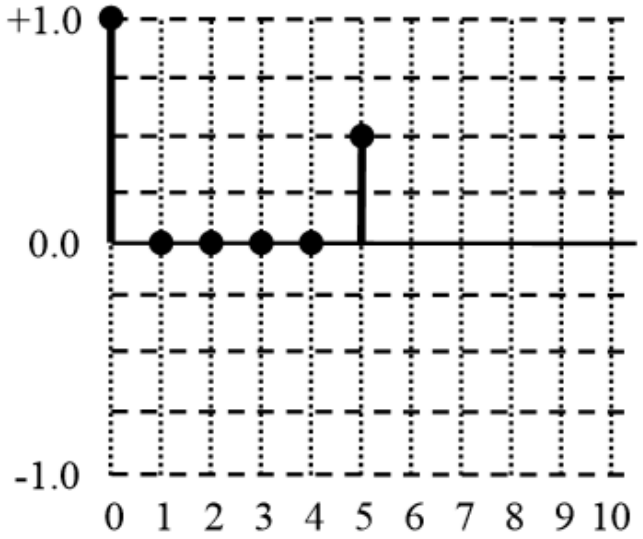


$y[n]$

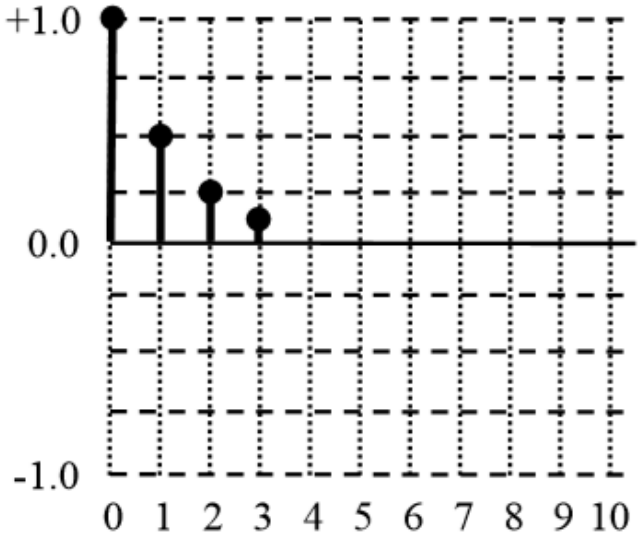
"non-overlapping echo" IR

Example convolutions:

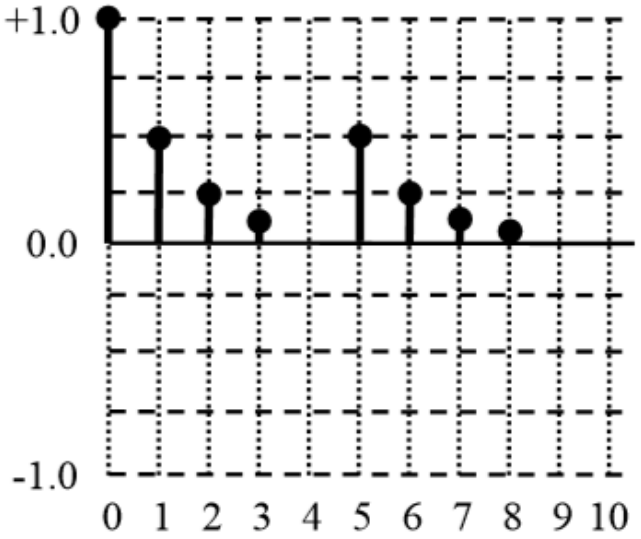
- Convolution is a commutative operation
  - That is: x[n] * h[n] = h[n] * x[n] = y[n]



$h[n]$ *



$x[n]$ =



$y[n]$

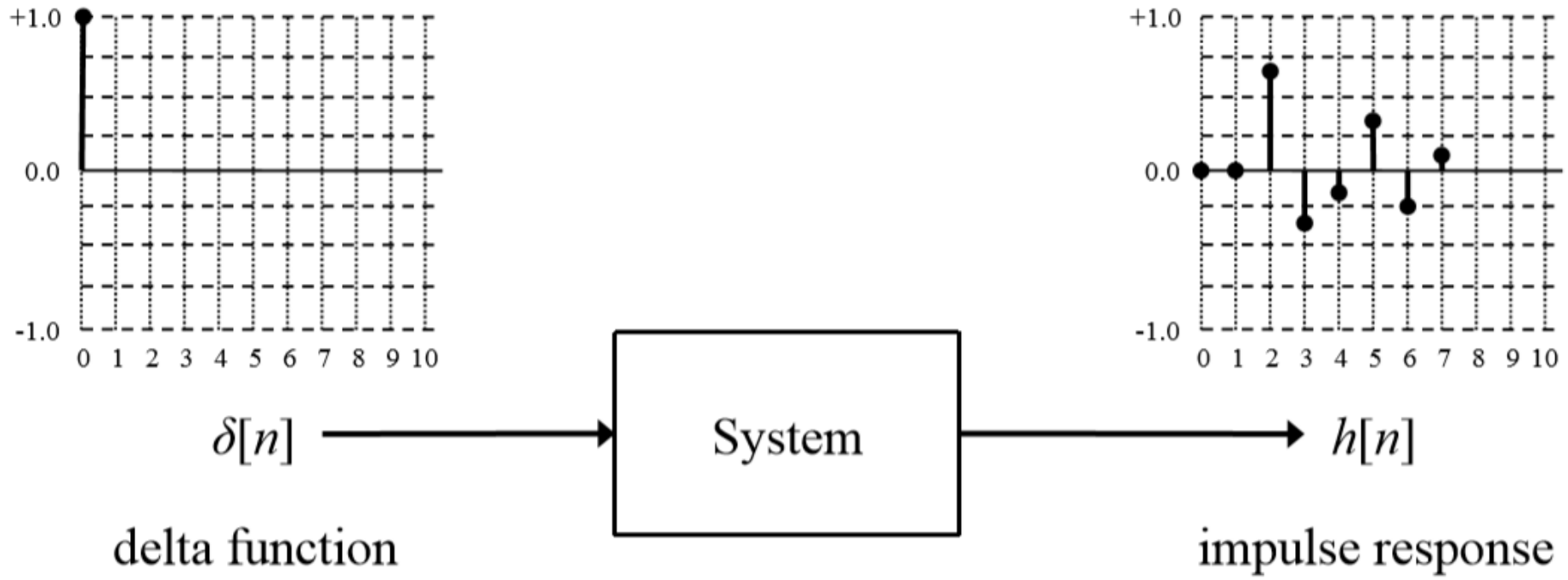
Time Domain Convolution

Time-Domain Convolution

- The signals x[n] and h[n] can be any arbitrary kind of signal
  - However, x[n] is usually an input signal that you want to process
  - And, h[n] is the impulse response (IR) of a system you want mode

Impulse Response

- The IR of a system is found by inputting a normalized impulse (delta function) into the system, and measuring or calculating the output

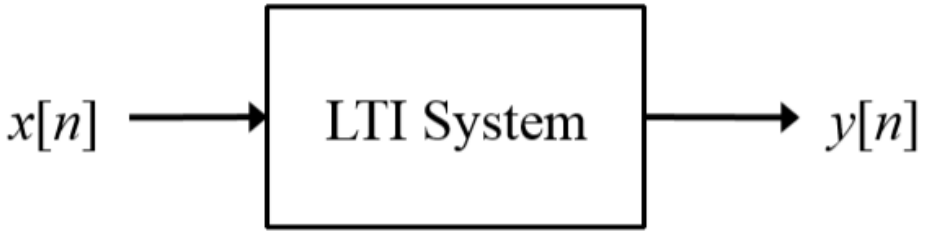


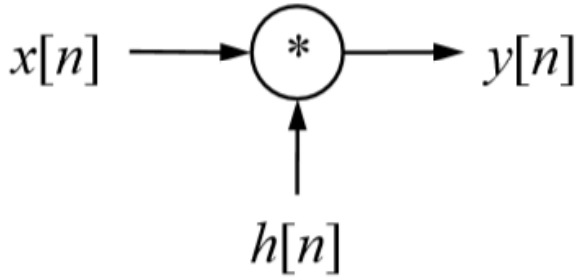
Model the System as it applied to signal change

- If the system is linear and time-invariant (LTI), the impulse response perfectly characterizes the system
  - That is, we know how the system will respond to any input it receives

Simulate the system as convolution

- We can thus simulate how a system would respond to an input signal by convolving the input signal with the system's IR
- That is, the following are equivalent:





Application: Concert Hall

Time-Domain Convolution

- The IR of a concert hall can be found by firing a starting pistol (an approximate delta function), and using a microphone to record to the system's response
  - Convolving a "dry" recording of an instrument with the IR results in an output signal where it sounds like the instrument is playing in the hall
    - This is the basis for a convolution reverb digital effect

Convolution using the FFT

- Time-domain convolution can be very slow, especially if h[n] is large
  - Is O(N × M), where N and M are the sizes of x[n] and h[n]
- High-speed or FFT convolution uses the principle that convolution in the time domain corresponds to multiplication of spectra in the frequency domain

Convolution using the FFT

- This is the convolution theorem, and is expressed mathematically as:

$$x[n] * h[n] \quad \underset{IDFT}{\overset{DFT}{\rightleftarrows}} \quad X[k] \cdot H[k]$$

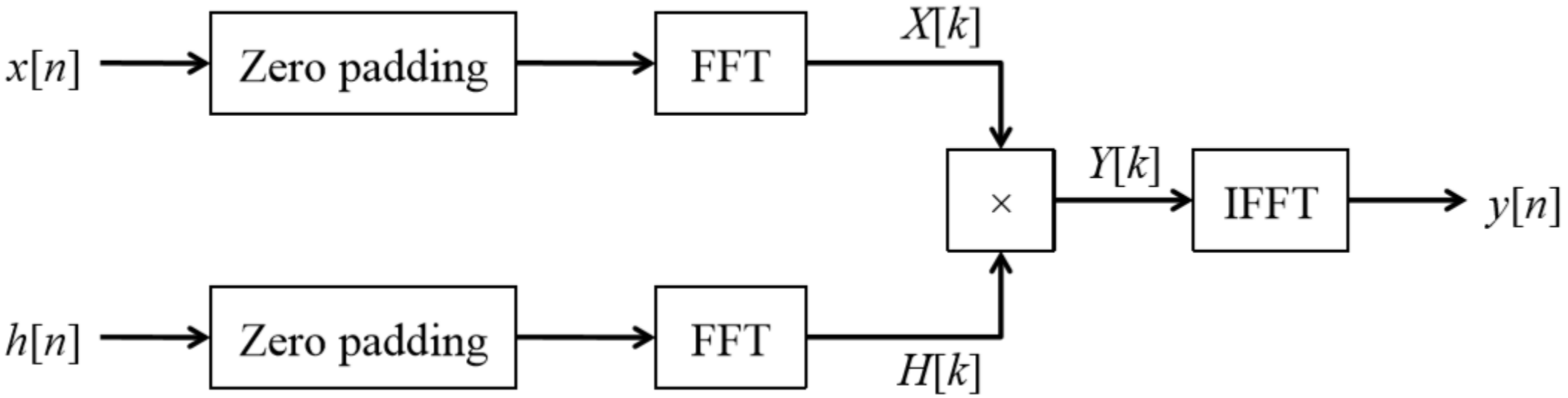
Time domain | Frequency domain

- Where X[k] and H[k] are the spectra of x[n] and h[n]:
  - X[k] = DFT[x[n]]
  - H[k] = DFT[h[n]

Convolution using the FFT

- We can calculate the convolution by:
  - Transforming both x[n] and h[n] to the frequency domain using the FFT
    - Note: zero-padding will have to be used so that x[n] and h[n] have the same length, the length is a power of 2 (needed for the FFT), and is long enough to avoid circular convolution (wrap around)
  - Multiplying X[k] by H[k] point by point
    - Note: this will be complex multiplication (imaginary/real)
  - Converting the result back to the time domain using the inverse FFT (IFFT)

Convolution using the FFT



Convolution using the FFT

- Note: the output from either the FFTs or the IFFT (but NOT both) will have to be scaled by dividing by each data point by N, if the FFT/IFFT algorithm you are using doesn't do scaling
- E.g.

```
for (k = 0, i = 0; k < N; k++, i += 2) {
        /*  Scale the real and imaginary parts of a data point */
        x[i] /= (double)N;
        x[i+1] /= (double)N;
}
```

Onward to …
next topic.

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https://pages.cpsc.ucalgary.ca/~hudsonj/

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