

Recursion

**CPSC 231: Introduction to Computer Science for Computer Science
Majors I
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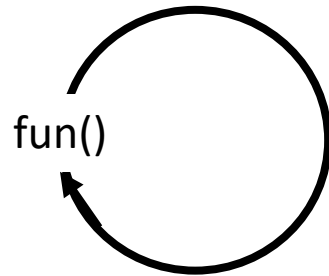
Recursion

- Definition:
 - See Recursion
 - Defining something in terms of itself
 - Generally using a smaller or simpler version
- Recursive Function
 - A function that calls itself

Recursion

- A programming technique whereby a function calls itself either directly or indirectly

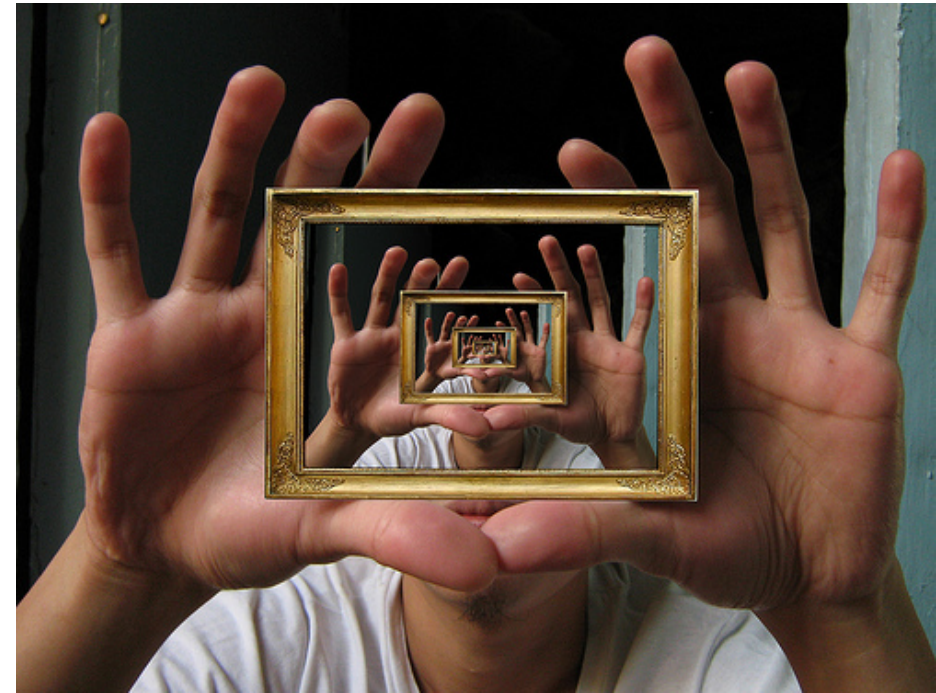
```
def fun ():  
    :  
    fun ()  
    :
```



```
def fun1 ():  
    fun2 ()  
  
def fun2 ():  
    fun1 ()
```

indirectly (mutual recursive)

Directly



Recursion

- A well-formed recursive function normally has two cases
 - Base Case:
 - Does not make a recursive call
 - Permits function to return (allowing us to fall out of the function calls)
 - Without a well-formed base case, you will have infinite recursive calls leading to the popular *StackOverflowException* or equivalently *RecursionError* in Python
 - Recursive Case:
 - Function calls itself
 - Generally must be a call to a smaller or simpler version of the problem

Tail Recursion

Tail Recursion

- a tail call is a subroutine call performed as the final action of a function (Ex. A final return statement)
- If a tail call calls the function itself, then this function is called tail recursive

Tail Recursion

- Typically when a function is called a record of memory (stack frame) is stored to track state from the previous function call
- But a tail recursive function doesn't need anything from before the final function call
- Modern compilers can identify this and reduce the process into an iterative loop for you. Although this is not guaranteed.
 - python is a procedural language, there is a class of languages called functional languages which will often guarantee this behaviour

Usage of Recursion

Useful Examples of Recursion

- Drawing fractals
- Finding a path through a maze
- Flood fill / “paint bucket” tool
- Merge sort, quick sort, binary search
- Finding the total size of all of the files in a directory and its subdirectories
- Parsing / evaluating expressions (how your code compiler knows you have syntax errors!)
- ...

Factorial

A Small Example - Factorial

Computing n factorial (n!):

- Defined as the result of multiplying all numbers from n to 1 for all $n > 0$. If $n == 0$, then result is 1.

$$n! = \begin{cases} 1 & n = 0 \\ n \times (n - 1)! & n > 0 \end{cases}$$

- $0! \rightarrow 1$
- $1! \rightarrow 1 * 1 = 1$
- $2! \rightarrow 2 * 1 * 1 = 2$
- $3! \rightarrow 3 * 2 * 1 * 1 = 6$

A Small Example - Factorial

Computing n factorial:

- Using a loop:
 - Initialize a variable named *answer* to 1 (the answer to 0!)
 - for i ranging from 2 to n (inclusive)
 - Multiply *answer* by i, storing the result back into *answer*
 - return *answer*

A Small Example - Factorial

Computing n factorial:

- Using a loop:
 - Initialize a variable named *answer* to 1 (the answer to 0!)
 - for i ranging from 2 to n (inclusive)
 - Multiply *answer* by i, storing the result back into *answer*
 - return *answer*
- Another solution comes from the definition

A Small Example

Recursive Definition!

- Another solution
 - By definition,
 - View

$$0! == 1$$

$$n! == n * (n-1)!$$

A Small Example

Recursive Definition!

- Another solution
 - BASE CASE,
 - RECURSIVE CASE,

$$0! == 1$$

$$n! == n * (n-1)!$$

Factorial - Memory

A Small Example - Factorial

```
def factorial(n):  
    if n == 0: #BASE CASE  
        return 1  
    return n * factorial(n-1) #RECURSIVE CASE
```

Note: code is easy, directly follows from recursive definition

How does it look like in memory?

A Small Example - Factorial

- Let's trace this code:

```
def factorial(n):  
    if n == 0: #BASE CASE  
        return 1  
    return n * factorial(n-1) #RECURSIVE CASE  
factorial(3)
```

A Small Example - Factorial - Walkthrough

Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    return n * factorial(n-1)  
factorial(3)
```

STACK

Trace:

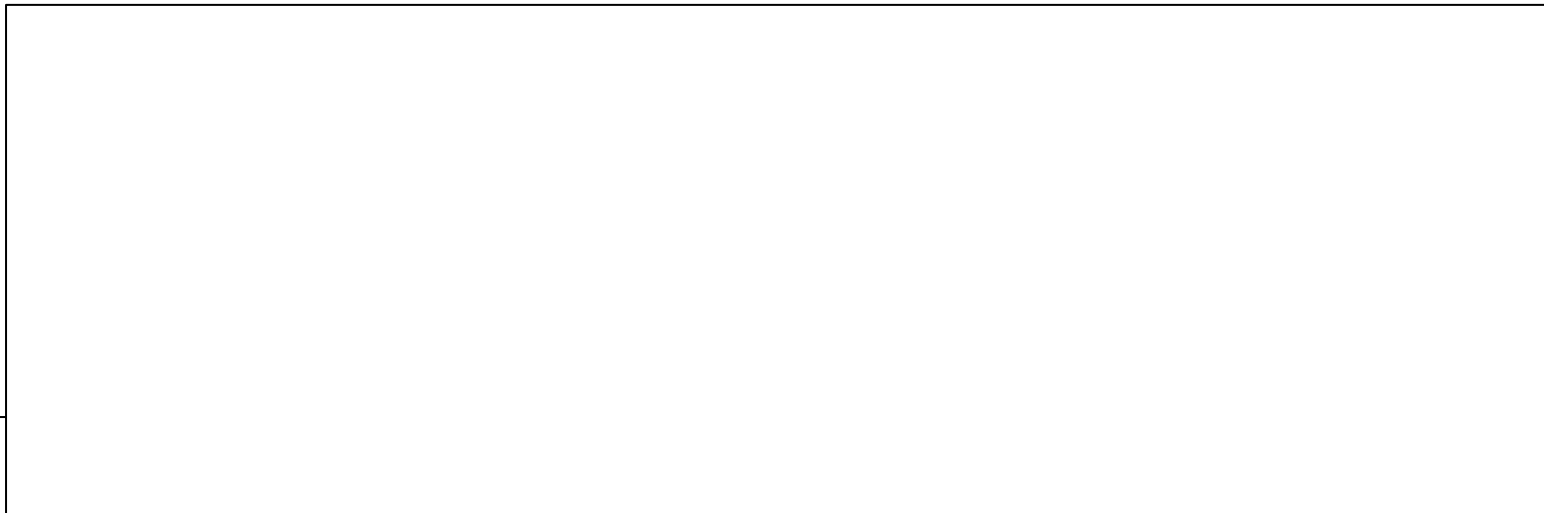
A Small Example - Factorial - Walkthrough

Code:

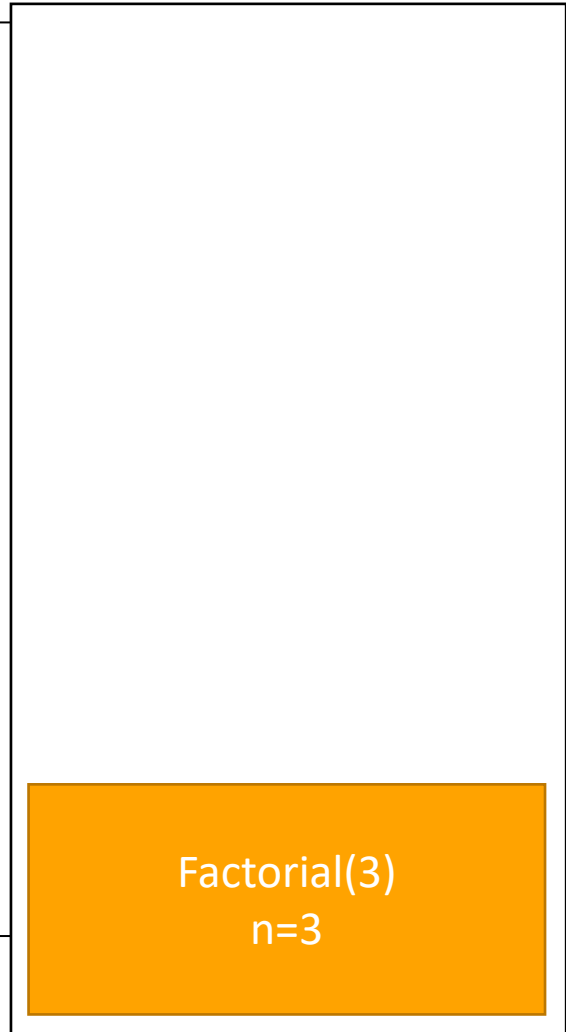
```
def factorial(n):  
    if n == 0:  
        return 1  
    return n * factorial(n-1)
```

 factorial(3)

Trace:



STACK



A Small Example - Factorial - Walkthrough

Code:

STACK

```
11 def factorial(n):  
    if n == 0:  
        return 1  
    return n * factorial(n-1)  
factorial(3)
```

Trace:

n=3

Factorial(3)
n=3

A Small Example - Factorial - Walkthrough

Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    return n * factorial(n-1)  
factorial(3)
```

Trace:

n=3

STACK

Factorial(3)
n=3

A Small Example - Factorial - Walkthrough

Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    return n * factorial(n-1)  
factorial(3)
```

FALSE



Trace:

n=3

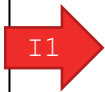
STACK

Factorial(3)
n=3

A Small Example - Factorial - Walkthrough

Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    return n * factorial(n-1)  
factorial(3)
```



Trace:

```
n=3  
return 3*(factorial(2))
```

STACK

Factorial(3)
n=3

A Small Example - Factorial - Walkthrough

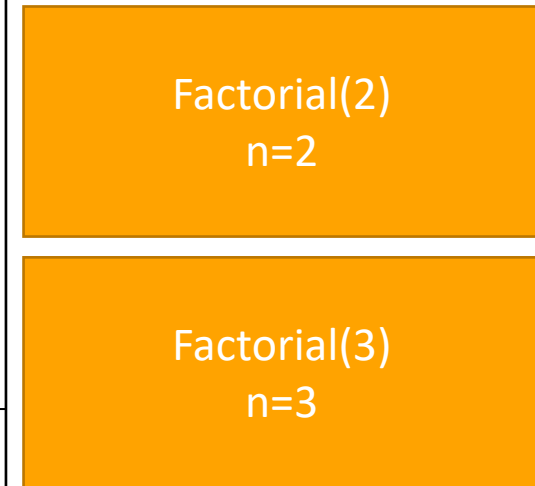
Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    return n * factorial(n-1)  
factorial(3)
```

Trace:

```
n=2  
return 3*(factorial(2))
```

STACK



A Small Example - Factorial - Walkthrough

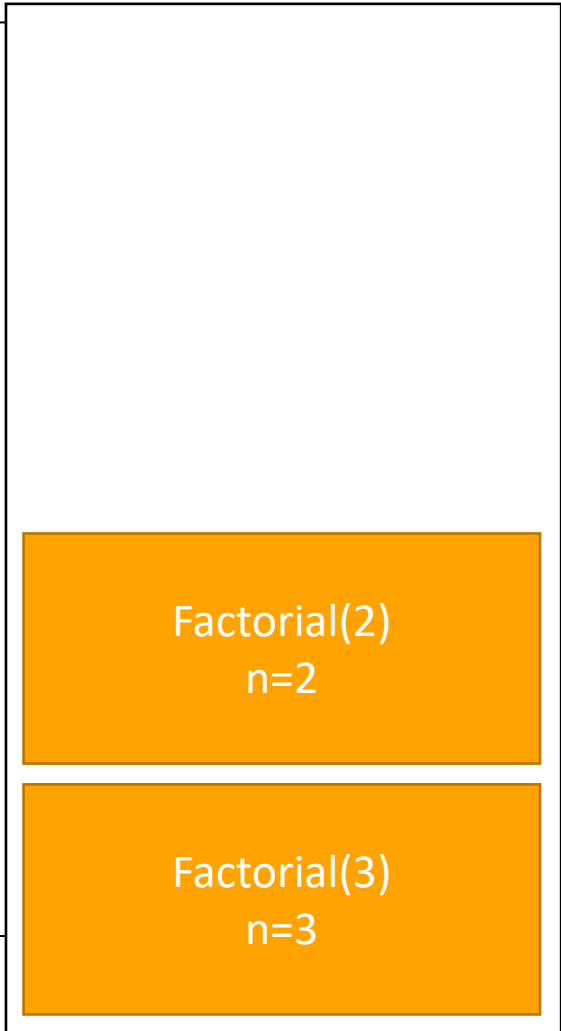
Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    return n * factorial(n-1)  
factorial(3)
```

Trace:

```
n=2  
return 3*(factorial(2))
```

STACK



A Small Example - Factorial - Walkthrough

Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    return n * factorial(n-1)  
factorial(3)
```

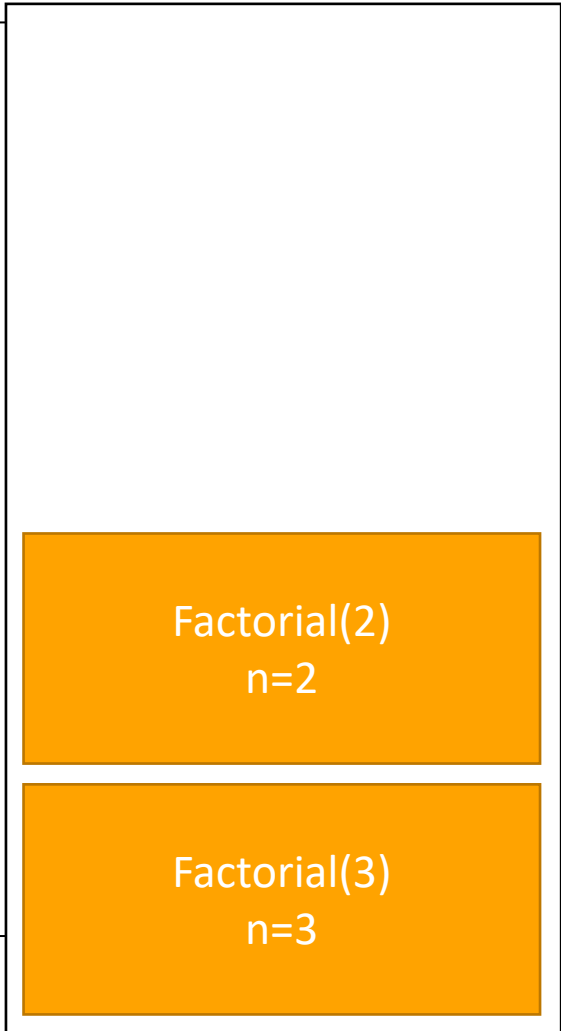
FALSE



Trace:

```
n=2  
return 3*(factorial(2))
```

STACK



A Small Example - Factorial - Walkthrough

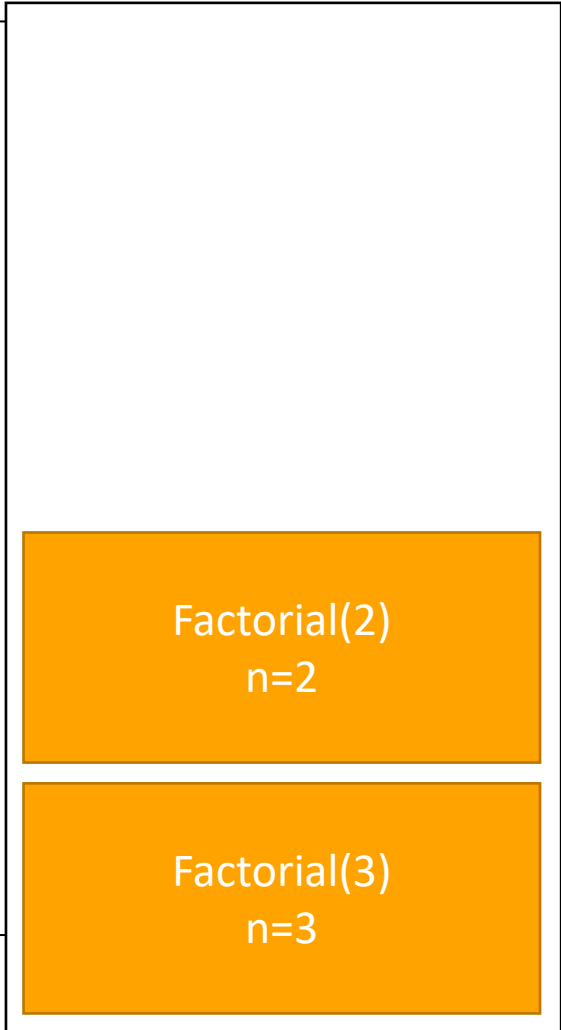
Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    12 → return n * factorial(n-1)  
factorial(3)
```

Trace:

```
n=2  
return 3*(2*factorial(1))
```

STACK



A Small Example - Factorial - Walkthrough

Code:

```
I3 → def factorial(n):  
    if n == 0:  
        return 1  
    I2 → return n * factorial(n-1)  
factorial(3)
```

Trace:

```
n=1  
return 3*(2*factorial(1))
```

STACK

Factorial(1)
n=1

Factorial(2)
n=2

Factorial(3)
n=3

A Small Example - Factorial - Walkthrough

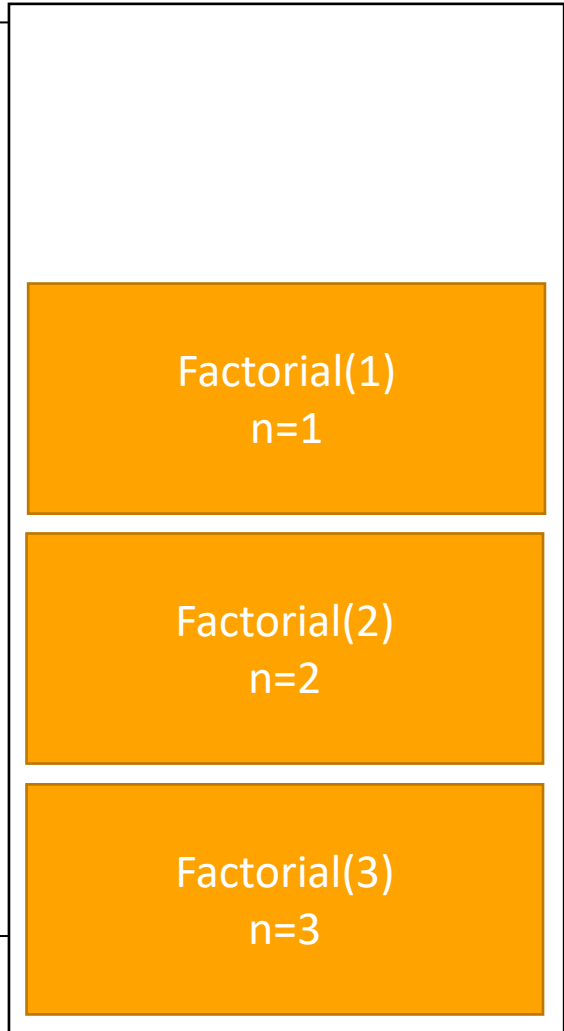
Code:

```
def factorial(n):  
    I3 → if n == 0:  
        return 1  
    I2 → return n * factorial(n-1)  
factorial(3)
```

Trace:

```
n=1  
return 3*(2*factorial(1))
```

STACK



A Small Example - Factorial - Walkthrough

Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    return n * factorial(n-1)  
factorial(3)
```

FALSE

I3 → if n == 0:

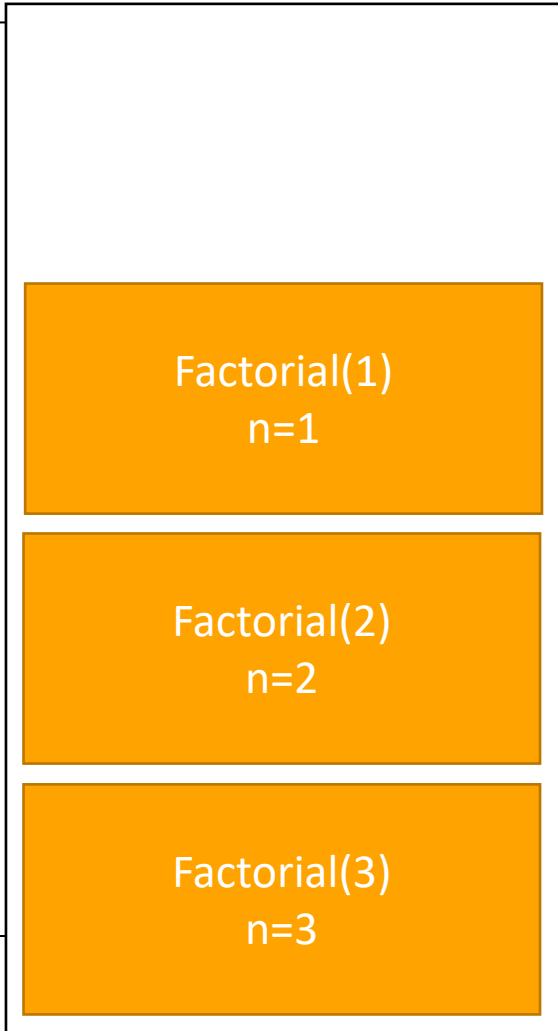
return 1

I2 → return n * factorial(n-1)

Trace:

```
n=1  
return 3 * (2 * factorial(1))
```

STACK



A Small Example - Factorial - Walkthrough

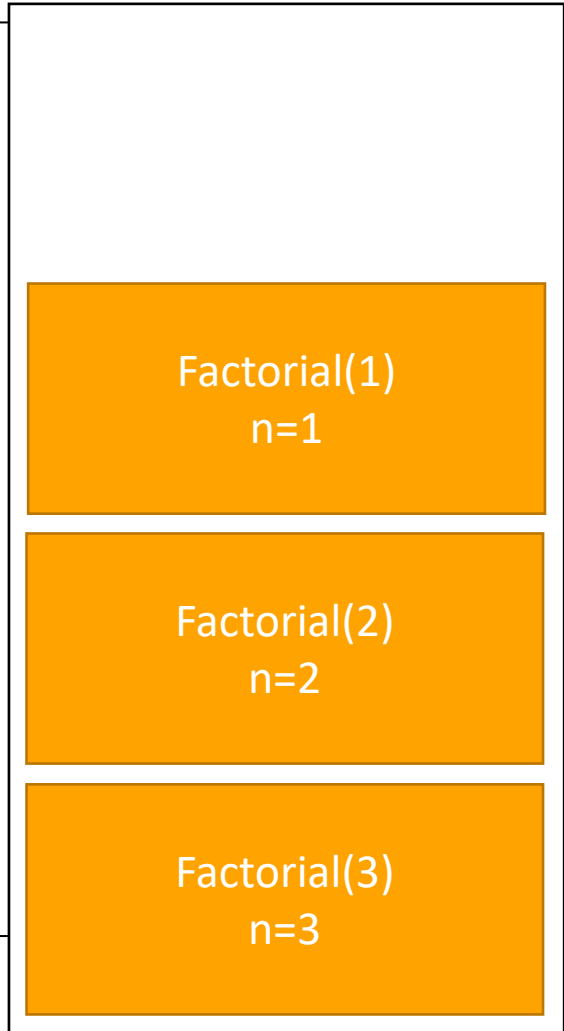
Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    I3 → return n * factorial(n-1)  
factorial(3)
```

Trace:

```
n=1  
return 3 * (2 * (1 * factorial(0)))
```

STACK



A Small Example - Factorial - Walkthrough

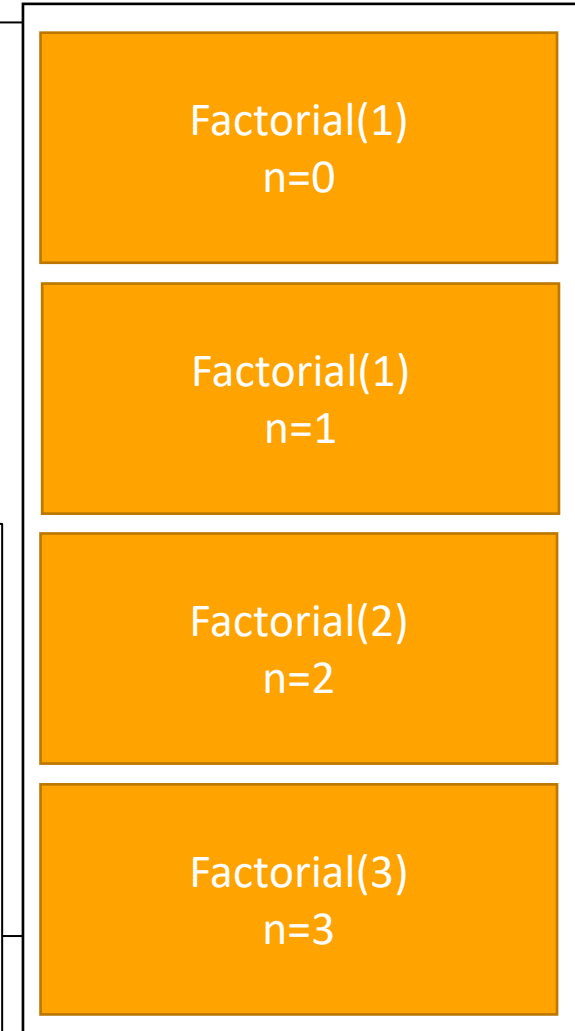
Code:

STACK

```
I4 → def factorial(n):  
    if n == 0:  
        return 1  
I3 → return n * factorial(n-1)  
factorial(3)
```

Trace:

```
n=0  
return 3*(2*(1*factorial(0)))
```



A Small Example - Factorial - Walkthrough

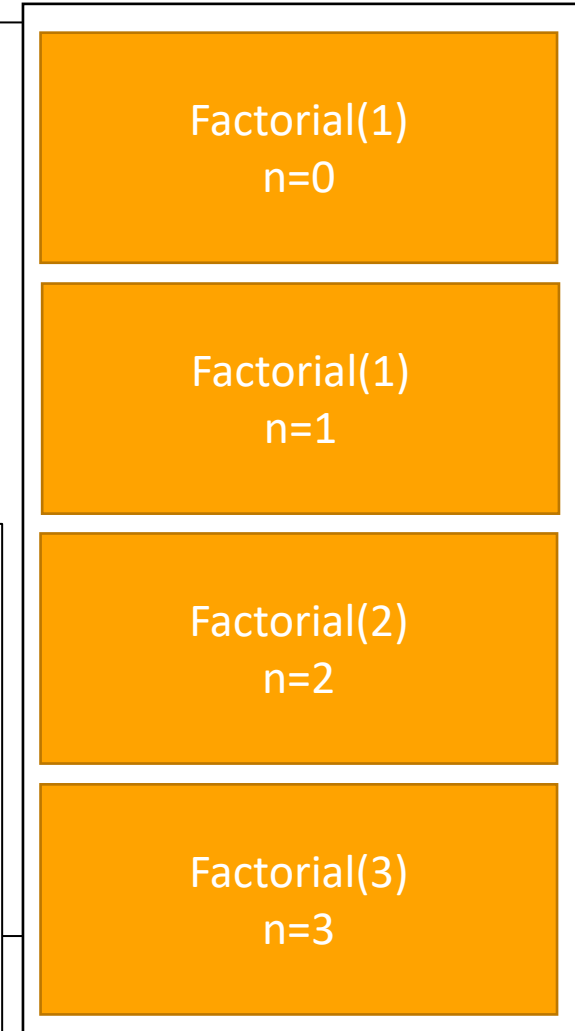
Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    return n * factorial(n-1)  
factorial(3)
```

Trace:

```
n=0  
return 3*(2*(1*factorial(0)))
```

STACK



A Small Example - Factorial - Walkthrough

Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    return n * factorial(n-1)  
factorial(3)
```

TRUE

I4

I3

Trace:

```
n=0  
return 3*(2*(1*factorial(0)))
```

STACK

Factorial(1)
n=0

Factorial(1)
n=1

Factorial(2)
n=2

Factorial(3)
n=3

A Small Example - Factorial - Walkthrough

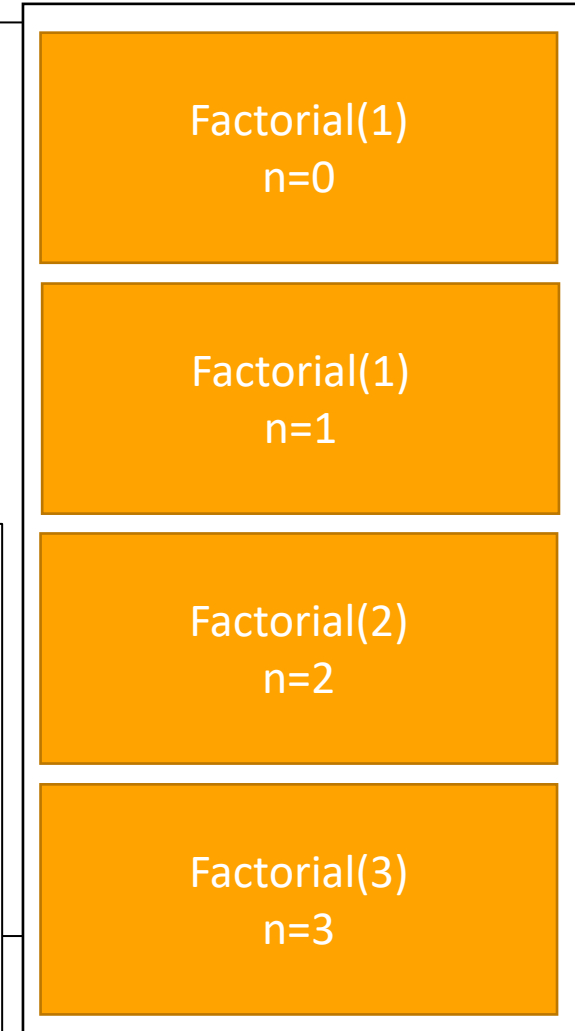
Code:

```
def factorial(n):  
    if n == 0:  
        I4 → return 1  
    I3 → return n * factorial(n-1)  
factorial(3)
```

Trace:

```
n=0  
return 3*(2*(1*factorial(0)))
```

STACK



A Small Example - Factorial - Walkthrough

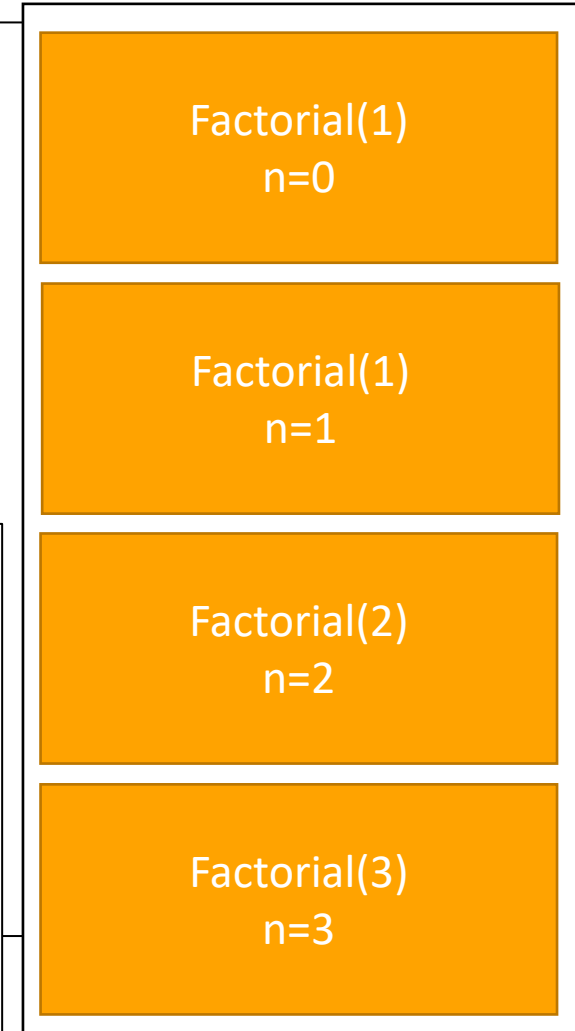
Code:

```
def factorial(n):  
    if n == 0:  
        I4 → return 1  
    I3 → return n * factorial(n-1)  
factorial(3)
```

Trace:

```
n=0  
return 3 * (2 * (1 * factorial(0)))  
                    └──────────┘  
                    factorial(0) return 1
```

STACK



A Small Example - Factorial - Walkthrough

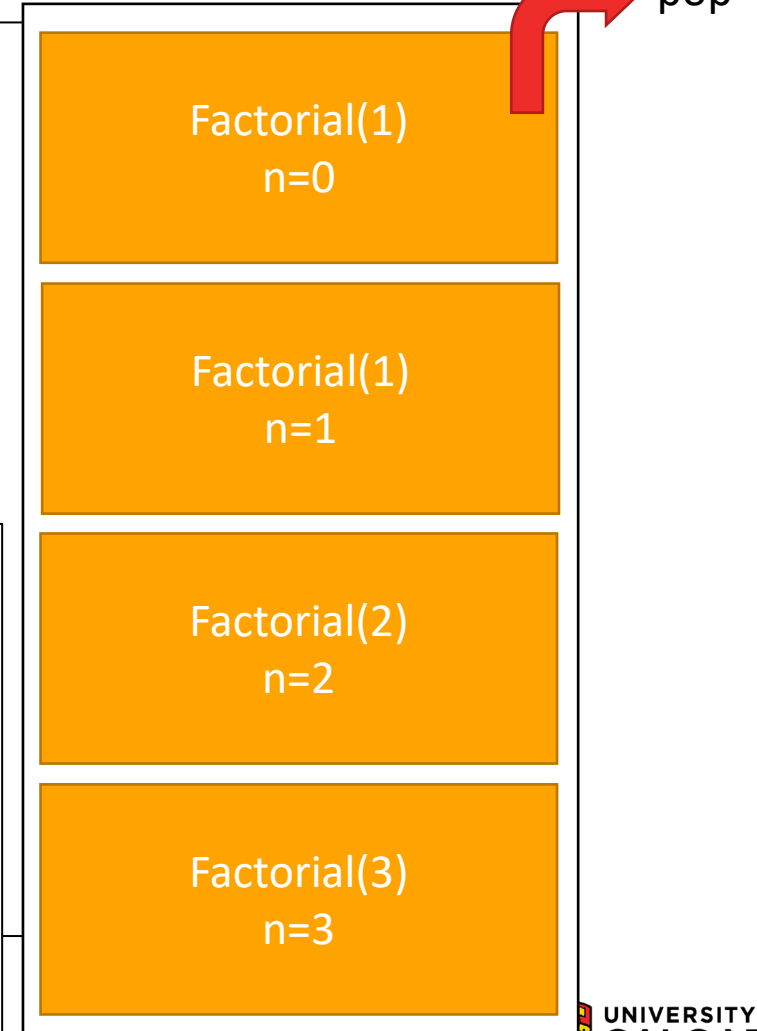
Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    I3 → return n * factorial(n-1)  
factorial(3)
```

Trace:

```
n=1  
return 3 * (2 * (1 * 1))
```

STACK



A Small Example - Factorial - Walkthrough

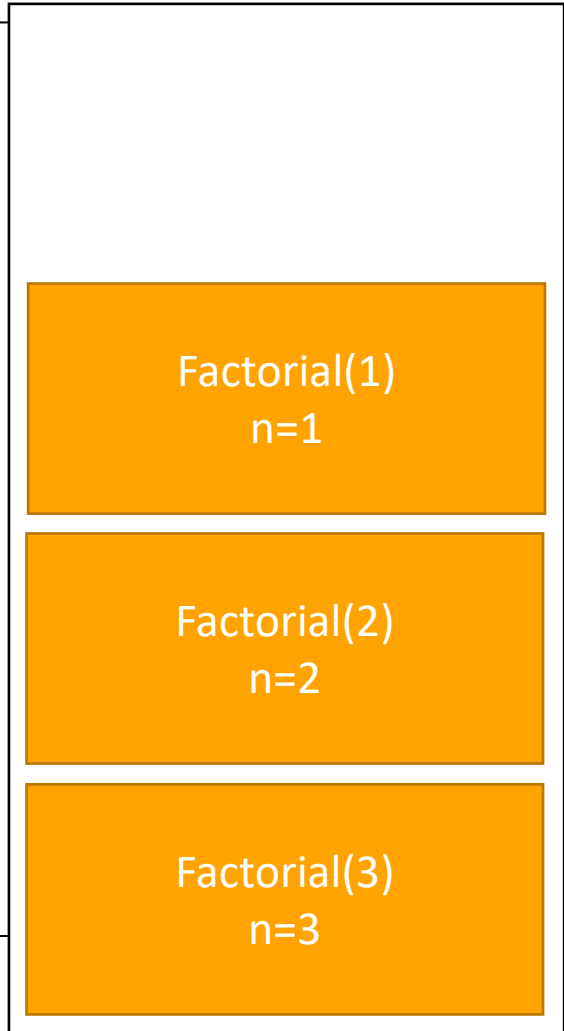
Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    I3 → return n * factorial(n-1)  
factorial(3)
```

Trace:

```
n=1  
return 3 * (2 * (1 * 1))
```

STACK



A Small Example - Factorial - Walkthrough

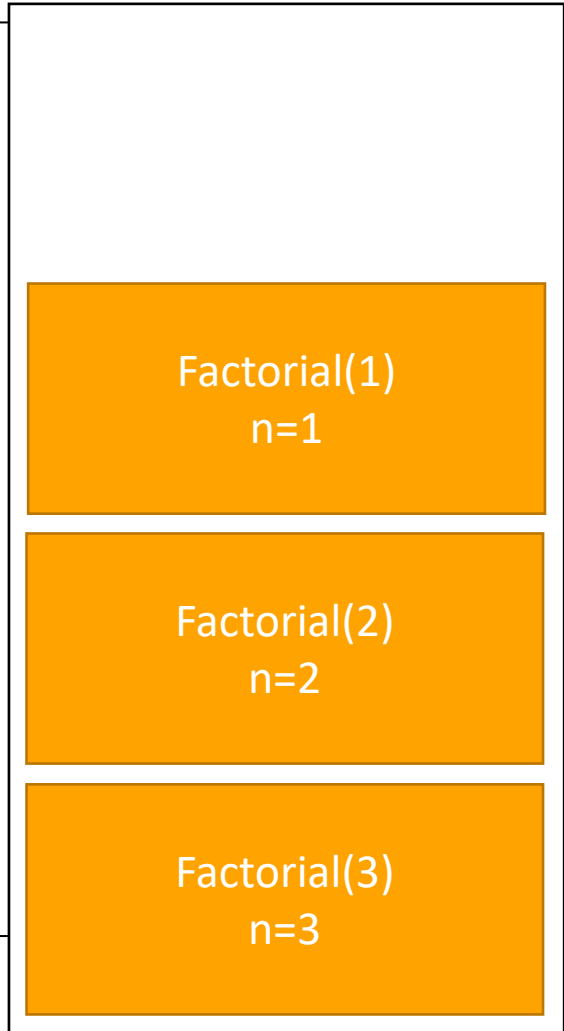
Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    ↪ return n * factorial(n-1)  
factorial(3)
```

Trace:

```
n=1  
return 3 * (2 * (1 * 1))  
                └──┬──  
                factorial(1) returns  
                1 * 1 = 1
```

STACK



A Small Example - Factorial - Walkthrough

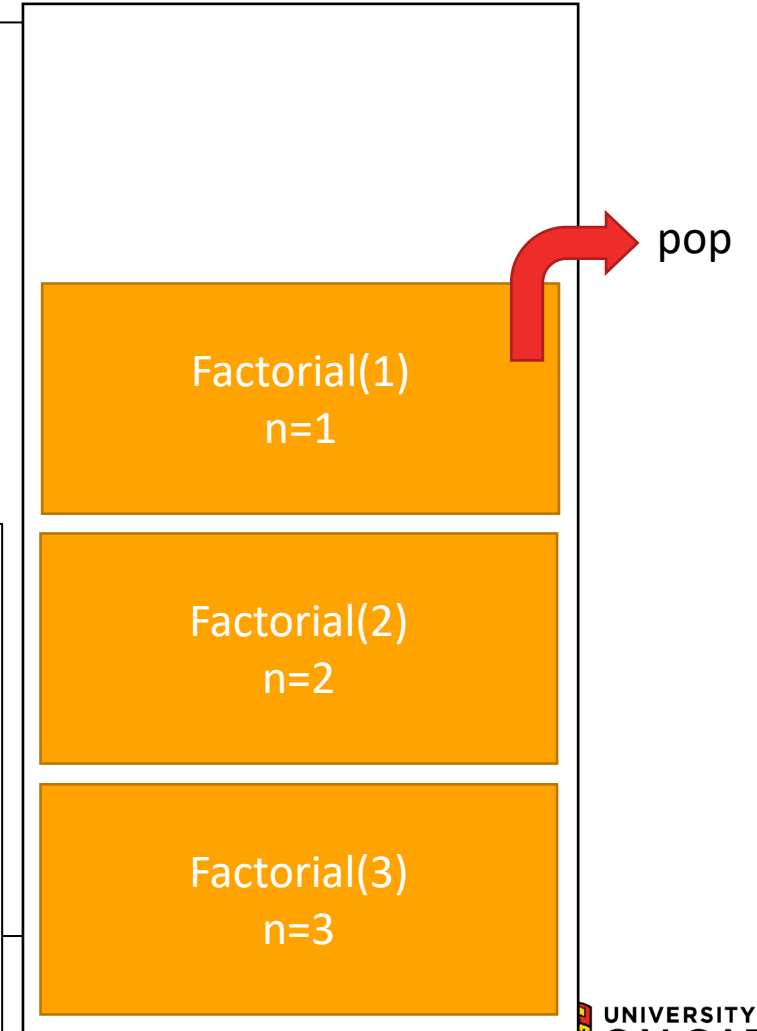
Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    I2 → return n * factorial(n-1)  
factorial(3)
```

Trace:

```
n=2  
return 3 * (2 * 1)
```

STACK



A Small Example - Factorial - Walkthrough

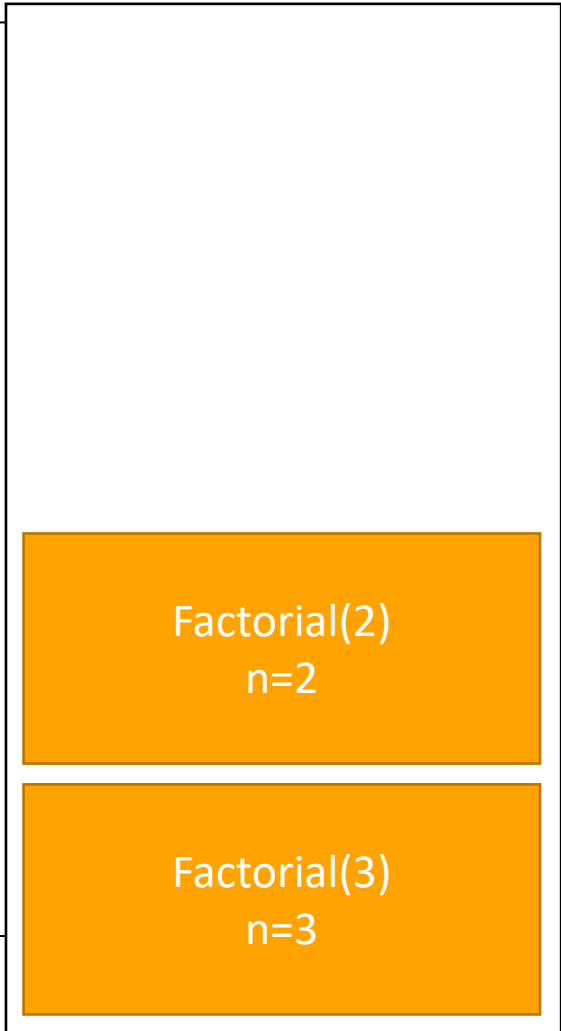
Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    I2 → return n * factorial(n-1)  
factorial(3)
```

Trace:

```
n=2  
return 3 * (2 * 1)
```

STACK



A Small Example - Factorial - Walkthrough

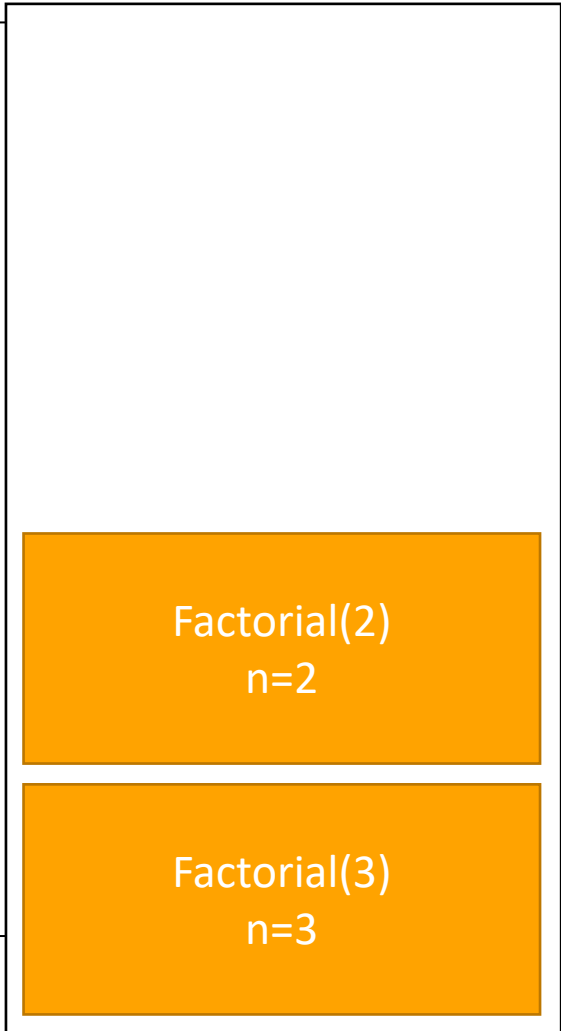
Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    I2 → return n * factorial(n-1)  
factorial(3)
```

Trace:

```
n=2  
return 3 * (2 * 1)  
           └─┬─┘  
           factorial(2) returns  
           2 * 1 = 2
```

STACK



A Small Example - Factorial - Walkthrough

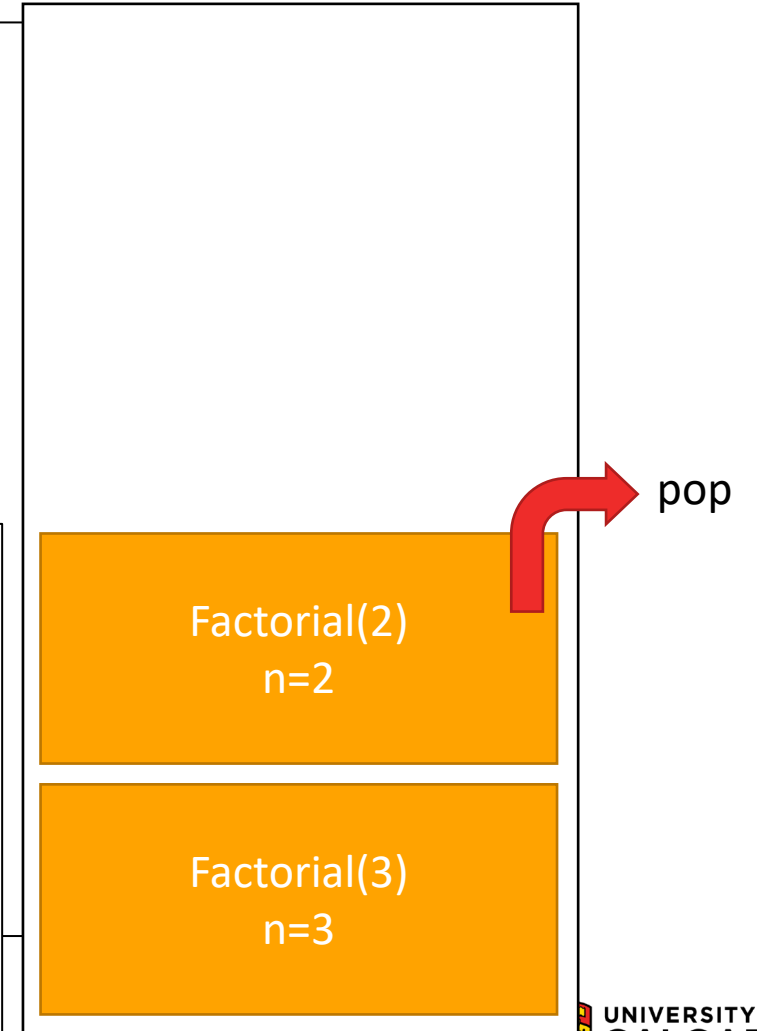
Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    11 return n * factorial(n-1)  
factorial(3)
```

Trace:

```
n=3  
return 3*2
```

STACK



A Small Example - Factorial - Walkthrough

Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    return n * factorial(n-1)  
factorial(3)
```



Trace:

```
n=3  
return 3*2
```

STACK

Factorial(3)
n=3


A Small Example - Factorial - Walkthrough

Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    return n * factorial(n-1)  
factorial(3)
```



Trace:

```
n=3  
return 3*2  
        
factorial(3) returns  
3*2 = 6
```

STACK



A Small Example - Factorial - Walkthrough

Code:

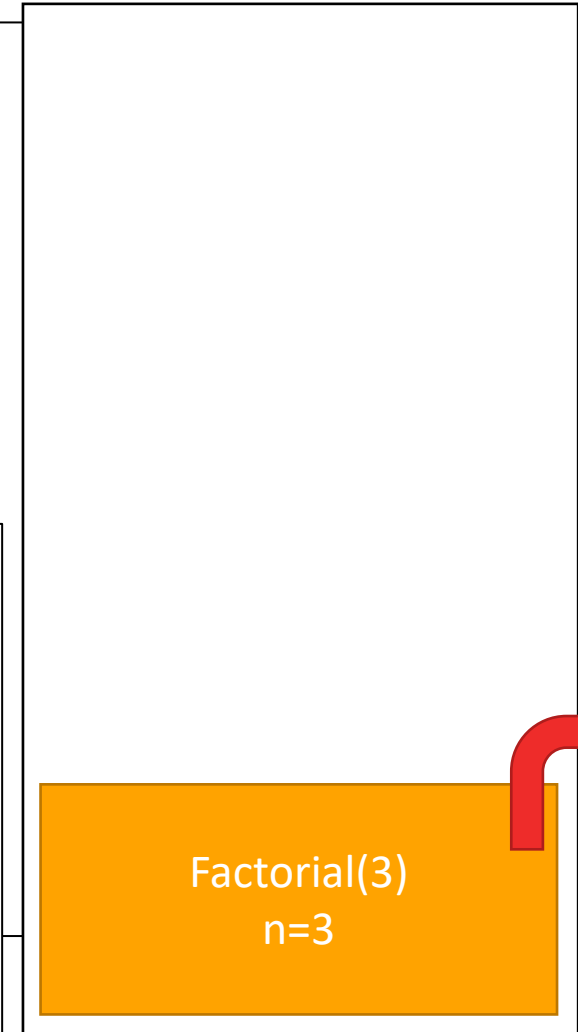
```
def factorial(n):  
    if n == 0:  
        return 1  
    return n * factorial(n-1)  
factorial(3)
```



Trace:

```
n=3  
return 6
```

STACK



A Small Example - Factorial - Walkthrough

Code:

```
def factorial(n):  
    if n == 0:  
        return 1  
    return n * factorial(n-1)  
factorial(3)
```



Trace:

6

STACK

Fibonacci

Fibonacci Numbers

- A sequence of values:
 - 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

Fibonacci Numbers

- A sequence of values:
 - 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...
- Defined recursively:
 - By definition:
 - $\text{fib}(0)$ is 0
 - $\text{fib}(1)$ is 1

Fibonacci Numbers

- A sequence of values:
 - 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...
- Defined recursively:
 - By definition:
 - fib(0) is 0
 - fib(1) is 1
 - Remaining values:
 - Formed by computing the sum of the previous two values in the sequence

Fibonacci Numbers

- A sequence of values:
 - 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...
- Defined recursively:
 - By definition:
 - fib(0) is 0
 - fib(1) is 1
 - Remaining values:
 - $\text{fib}(n) = \text{fib}(n-1) + \text{fib}(n-2)$

Fibonacci Numbers

Note that there are two base cases here!

Eg. fib(4)

$$\text{fib}(4) = \text{fib}(3) + \text{fib}(2)$$

$$\text{fib}(3) = \text{fib}(2) + \text{fib}(1)$$

$$\text{fib}(2) = \text{fib}(1) + \text{fib}(0)$$

$$\text{fib}(1) = 1$$

$$\text{fib}(0) = 0$$

Fibonacci Numbers

Note that there are two base cases here!

Eg. fib(4)

$$\text{fib}(4) = \text{fib}(3) + \text{fib}(2) = \text{fib}(1) + \text{fib}(0) + \text{fib}(1) + \text{fib}(1) + \text{fib}(0) = 3$$

$$\text{fib}(3) = \text{fib}(2) + \text{fib}(1) = \text{fib}(1) + \text{fib}(0) + \text{fib}(1) = 2$$

$$\text{fib}(2) = \text{fib}(1) + \text{fib}(0) = 1$$

$$\text{fib}(1) = 1$$

$$\text{fib}(0) = 0$$

Example - Fibonacci Numbers

Note that there are two base cases here!

Eg. fib(4)

$$\text{fib}(4) = \text{fib}(3) + \text{fib}(2) = \underbrace{\text{fib}(1) + \text{fib}(0) + \text{fib}(1)}_{\text{fib}(2)} + \underbrace{\text{fib}(1) + \text{fib}(0)}_{\text{fib}(2)} = 3$$

$$\text{fib}(3) = \text{fib}(2) + \text{fib}(1) = \text{fib}(1) + \text{fib}(0) + \text{fib}(1) = 2$$

$$\text{fib}(2) = \text{fib}(1) + \text{fib}(0) = 1$$

$$\text{fib}(1) = 1$$

$$\text{fib}(0) = 0$$

Fibonacci Numbers

Note that there are two base cases here!

Eg. fib(5)

$$\text{fib}(5) = \text{fib}(4) + \text{fib}(3) = \text{fib}(1) + \text{fib}(0) + \text{fib}(1) + \text{fib}(1) + \text{fib}(0) + \text{fib}(1) + \text{fib}(0) + \text{fib}(1) = 5$$

$$\text{fib}(4) = \text{fib}(3) + \text{fib}(2) = \text{fib}(1) + \text{fib}(0) + \text{fib}(1) + \text{fib}(1) + \text{fib}(0) = 3$$

$$\text{fib}(3) = \text{fib}(2) + \text{fib}(1) = \text{fib}(1) + \text{fib}(0) + \text{fib}(1) = 2$$

$$\text{fib}(2) = \text{fib}(1) + \text{fib}(0) = 1$$

$$\text{fib}(1) = 1$$

$$\text{fib}(0) = 0$$

Example - Fibonacci Numbers

Note that there are two base cases here!

Eg. fib(5)

$$\text{fib}(5) = \text{fib}(4) + \text{fib}(3) = \overbrace{\text{fib}(1) + \text{fib}(0) + \text{fib}(1) + \text{fib}(1) + \text{fib}(0)}^{\text{fib}(4)} + \overbrace{\text{fib}(1) + \text{fib}(0) + \text{fib}(1)}^{\text{fib}(3)} = 5$$

$$\text{fib}(4) = \text{fib}(3) + \text{fib}(2) = \text{fib}(1) + \text{fib}(0) + \text{fib}(1) + \text{fib}(1) + \text{fib}(0) = 3$$

$$\text{fib}(3) = \text{fib}(2) + \text{fib}(1) = \text{fib}(1) + \text{fib}(0) + \text{fib}(1) = 2$$

$$\text{fib}(2) = \text{fib}(1) + \text{fib}(0) = 1$$

$$\text{fib}(1) = 1$$

$$\text{fib}(0) = 0$$

Fibonacci Numbers

```
def fibRec(n):  
    if n == 0:  
        return 0  
    if n == 1:  
        return 1  
    return fibRec(n-1) + fibRec(n-2)
```

Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

fib(4)

fib(4)

Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

$fib(4) \rightarrow fib(3) + fib(2)$

fib(4)

Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

fib(4) → **fib(3)** + fib(2)

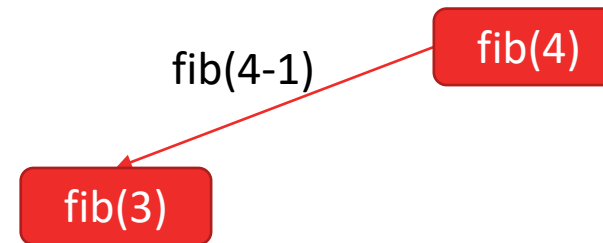
The expression is evaluated left to right.
So, fib(3) is calculated first.

fib(4)

Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

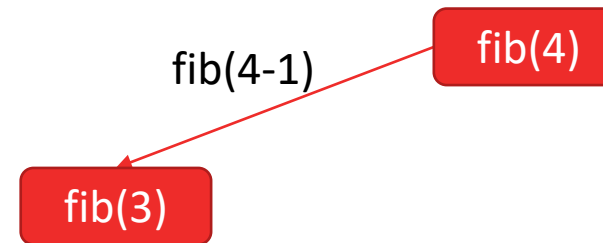
fib(3)



Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

$fib(3) \rightarrow fib(2) + fib(1)$

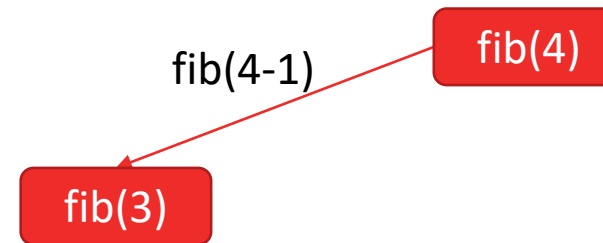


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{cases}$$

$fib(3) \rightarrow \mathbf{fib(2)} + fib(1)$

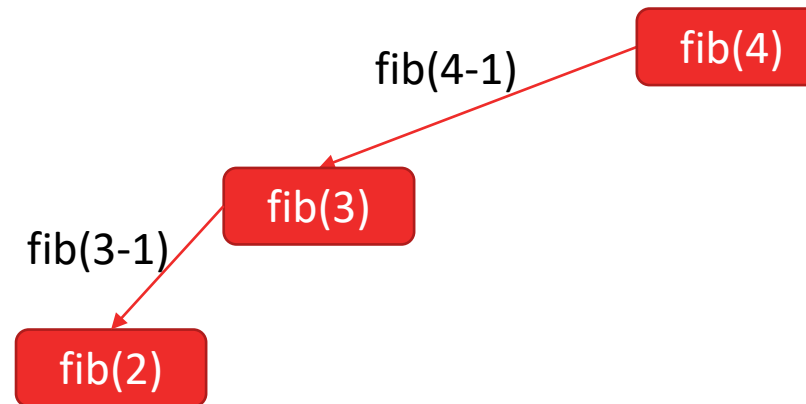
Similarly, $fib(2)$ is evaluated first.



Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

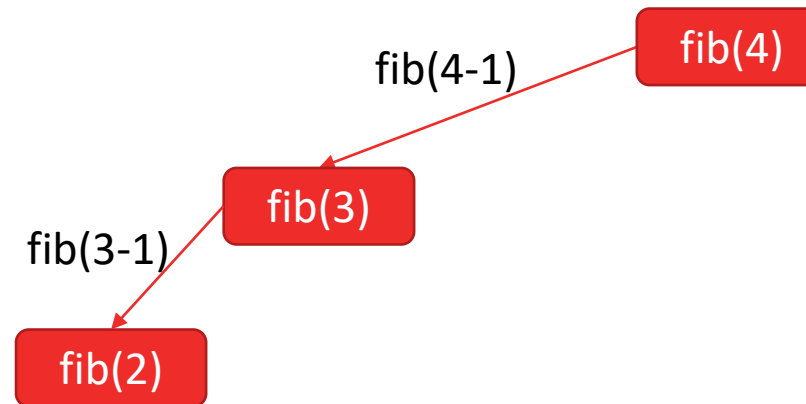
fib(2)



Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

$fib(2) \rightarrow fib(1) + fib(0)$

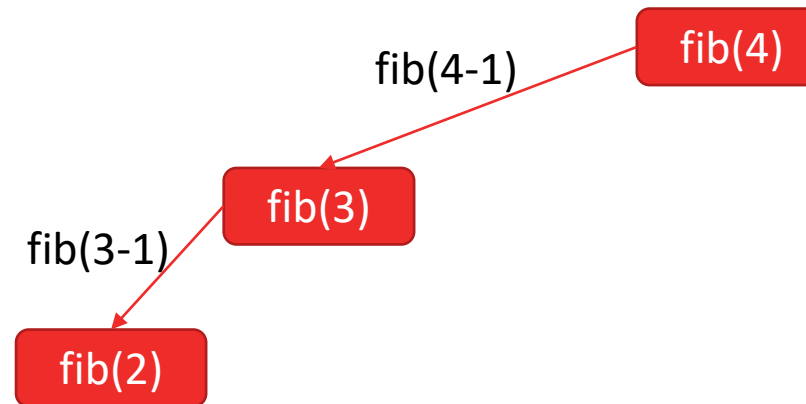


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{cases}$$

$fib(2) \rightarrow fib(1) + fib(0)$

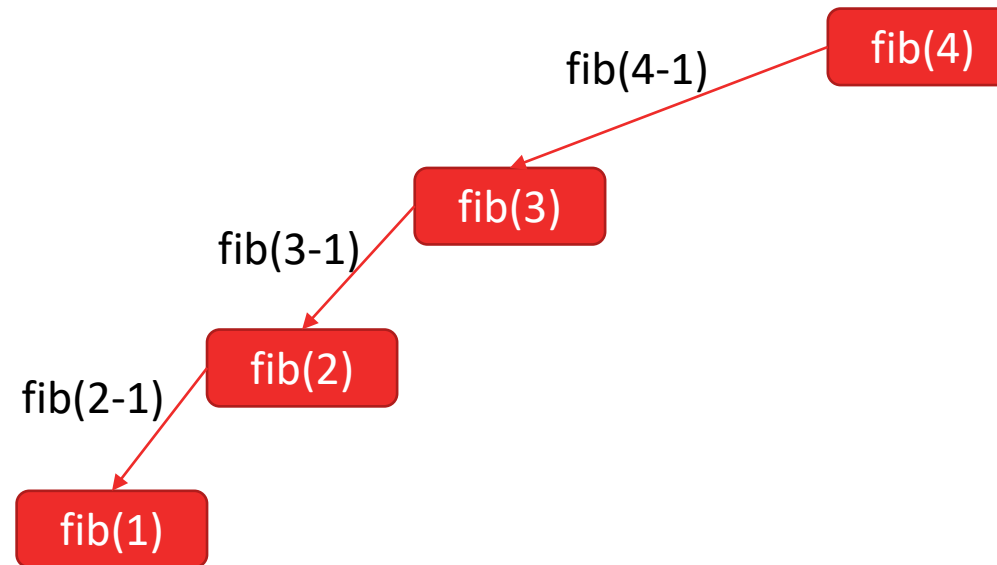
$fib(1)$ is evaluated first



Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

fib(1)

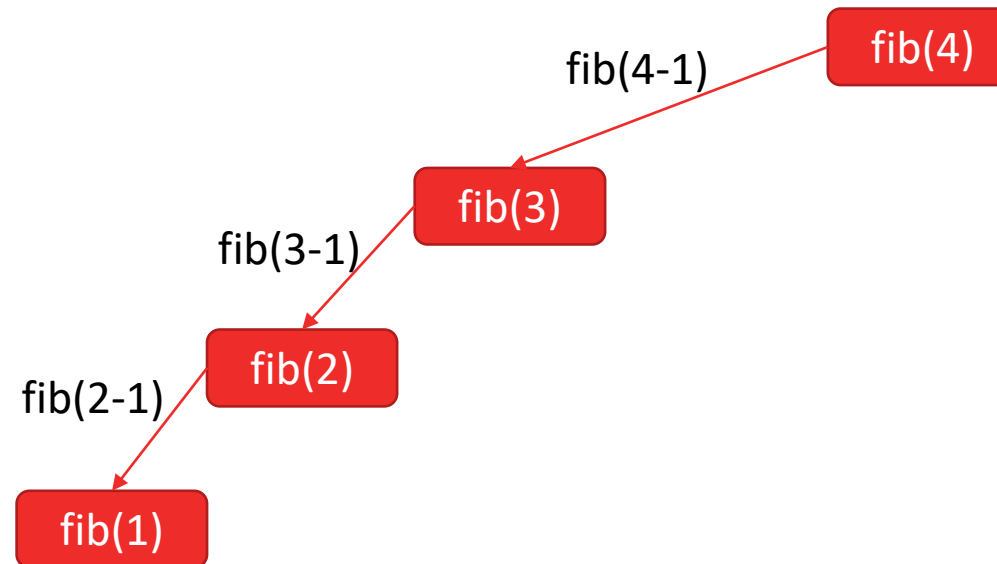


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{cases}$$

fib(1) → 1

fib(1) returns 1 to caller

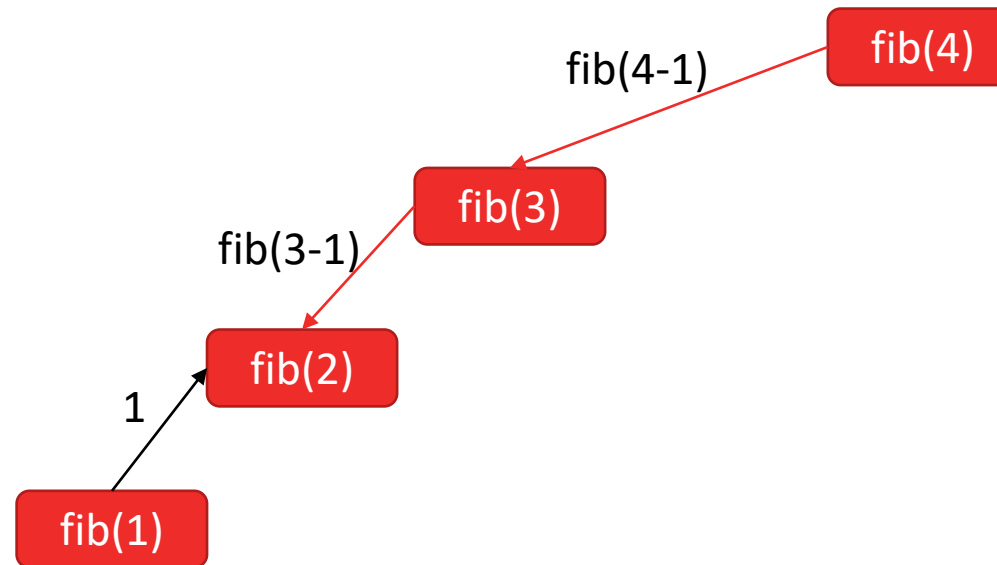


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

$fib(1) \rightarrow 1$

$fib(1)$ returns 1 to caller

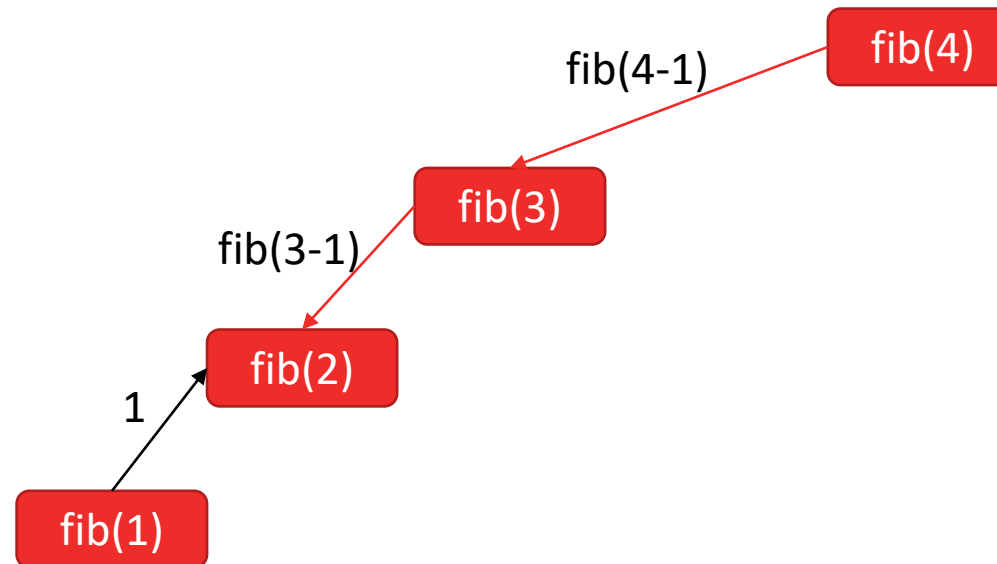


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{cases}$$

$fib(2) \rightarrow 1 + \mathbf{fib(0)}$

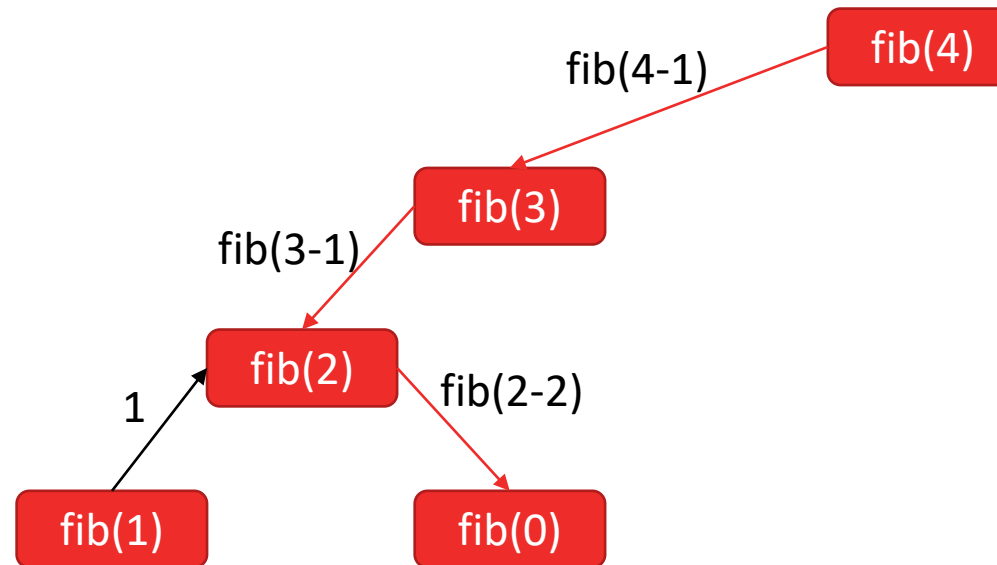
Now, $fib(0)$ is evaluated.



Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{cases}$$

fib(0)

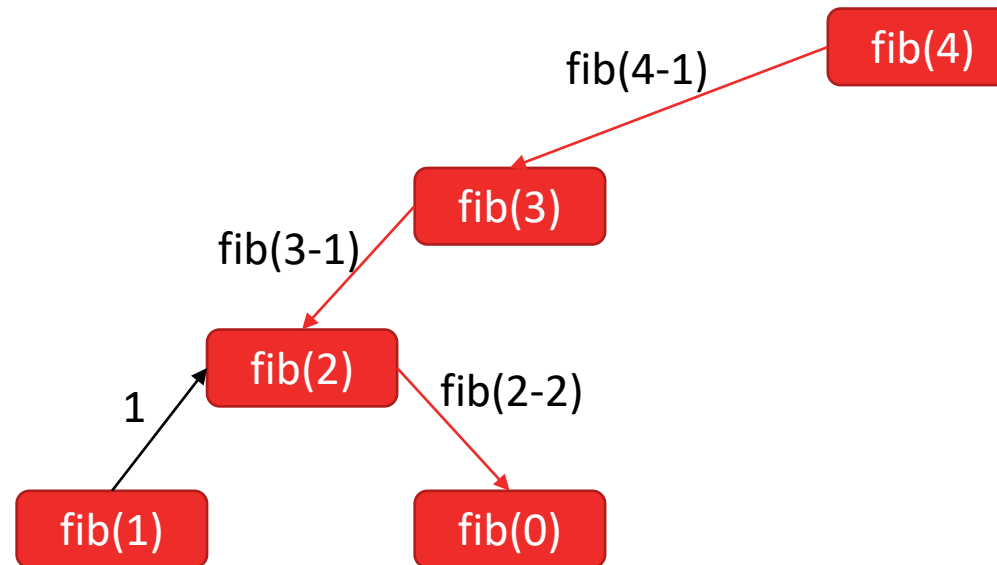


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{cases}$$

$fib(0) \rightarrow 0$

$fib(0)$ returns 0 to caller

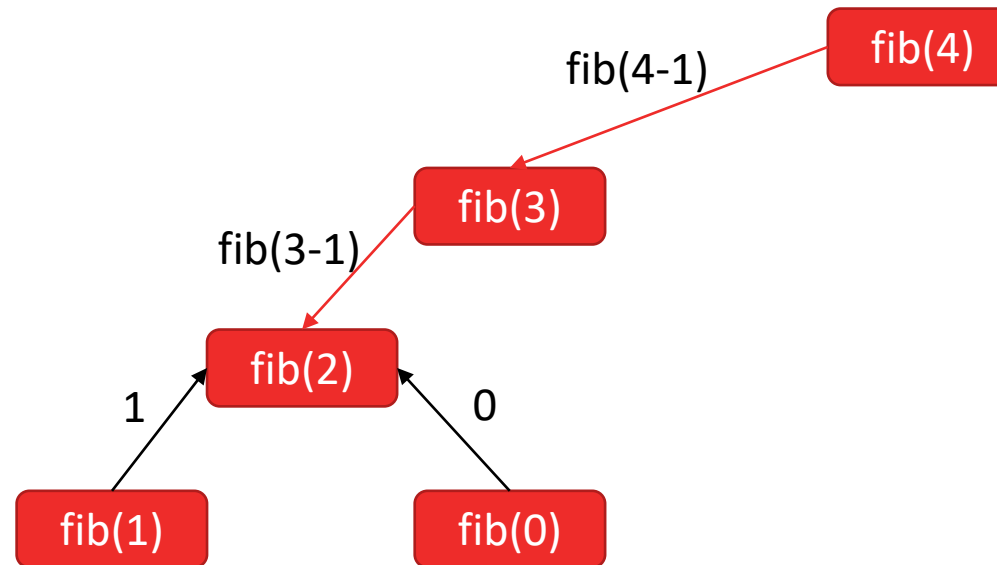


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

$fib(0) \rightarrow 0$

$fib(0)$ returns 0 to caller

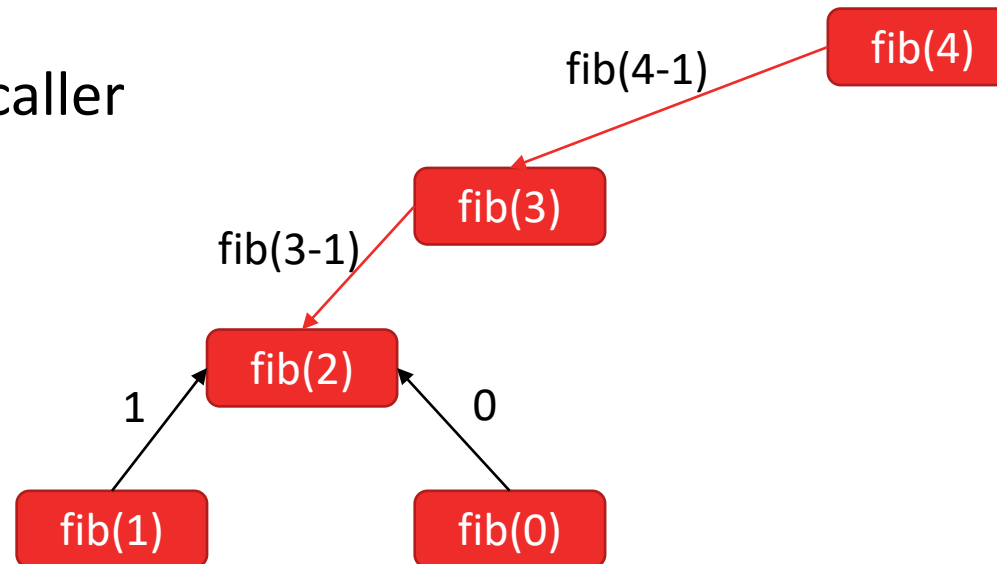


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{cases}$$

$fib(2) \rightarrow 1+0 = 1$

Now, $fib(2)$ returns 1 to the caller

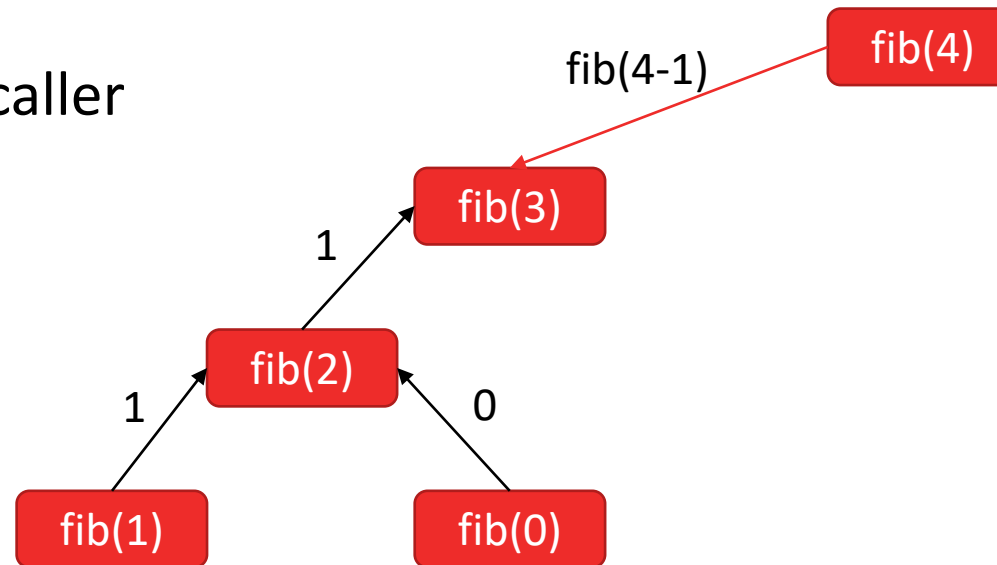


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

$fib(2) \rightarrow 1+0 = 1$

Now, $fib(2)$ returns 1 to the caller

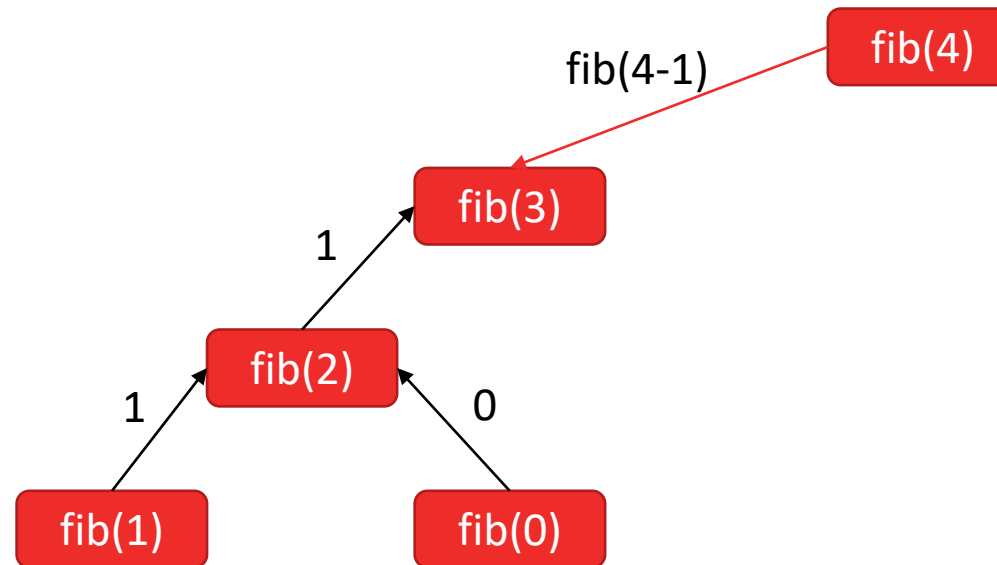


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

$fib(3) \rightarrow 1 + \mathbf{fib(1)}$

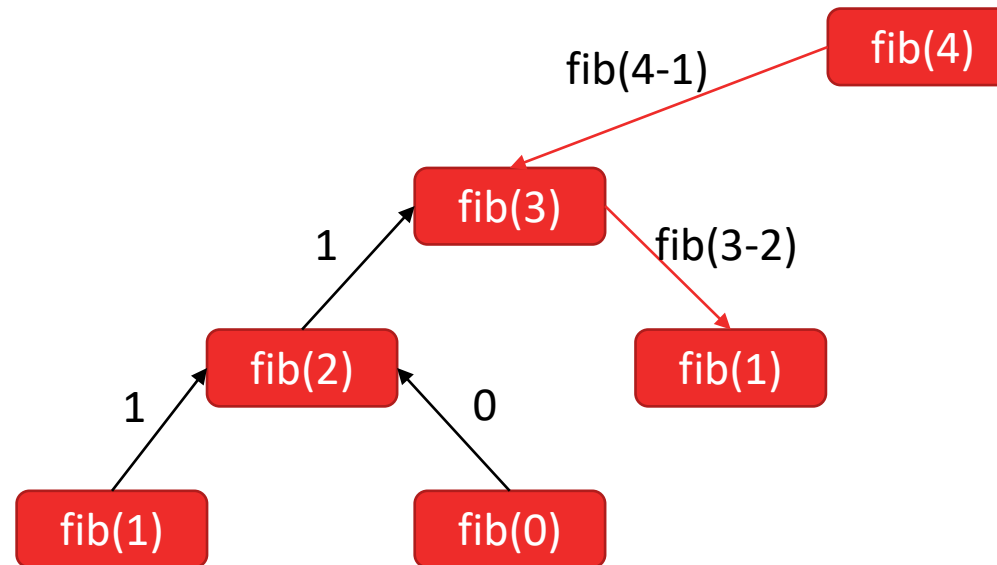
Now, $fib(1)$ is calculated



Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{cases}$$

fib(1)

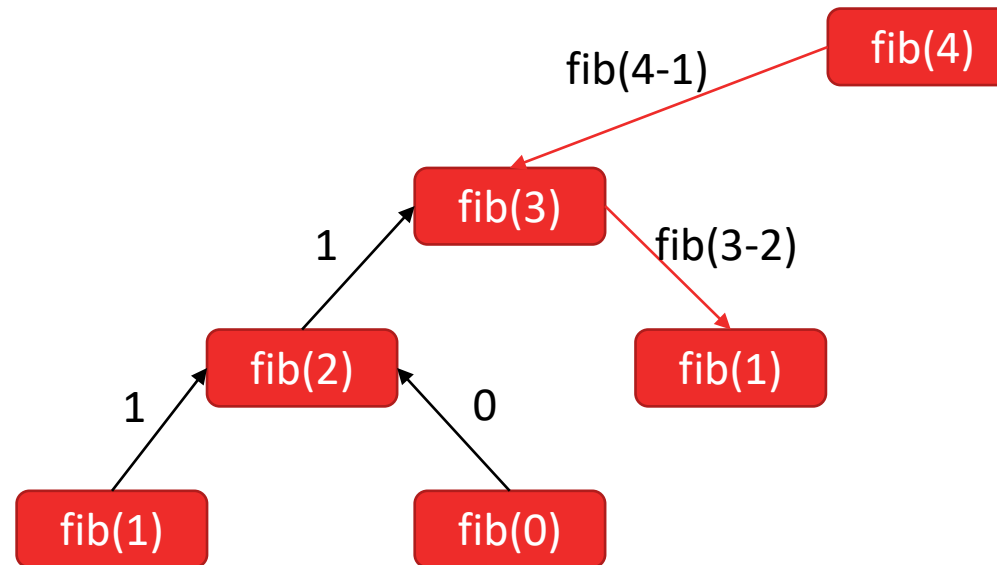


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{cases}$$

$fib(1) \rightarrow 1$

$fib(1)$ returns 1 to the caller.

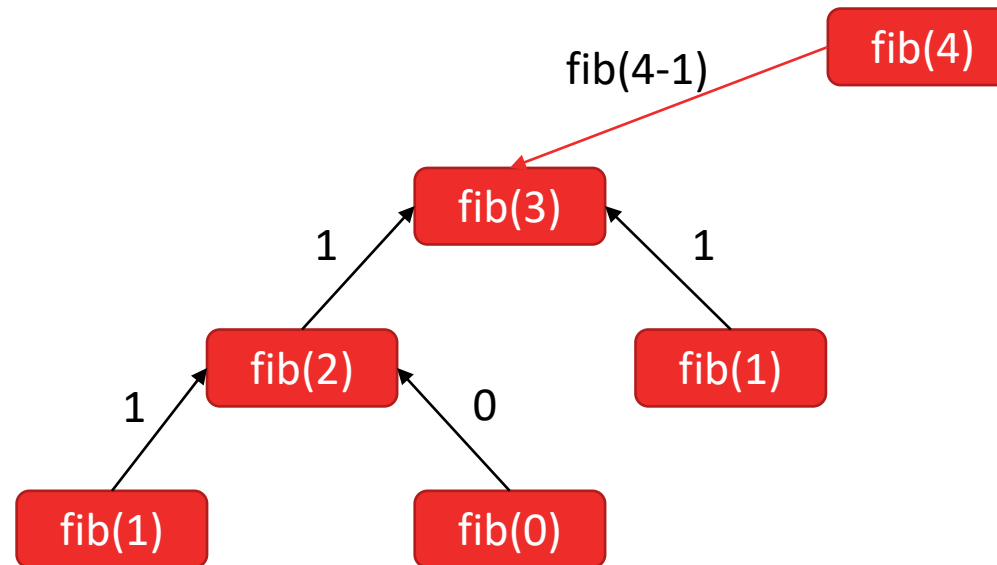


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{cases}$$

fib(1) → 1

fib(1) returns 1 to the caller.

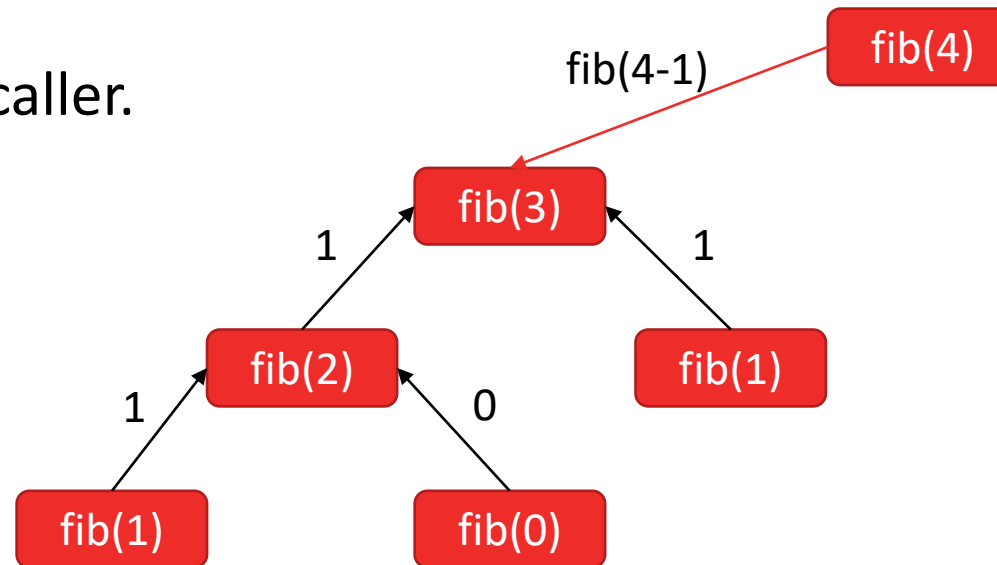


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

$fib(3) \rightarrow 1 + 1 = 2$

Now, $fib(3)$ returns 2 to the caller.

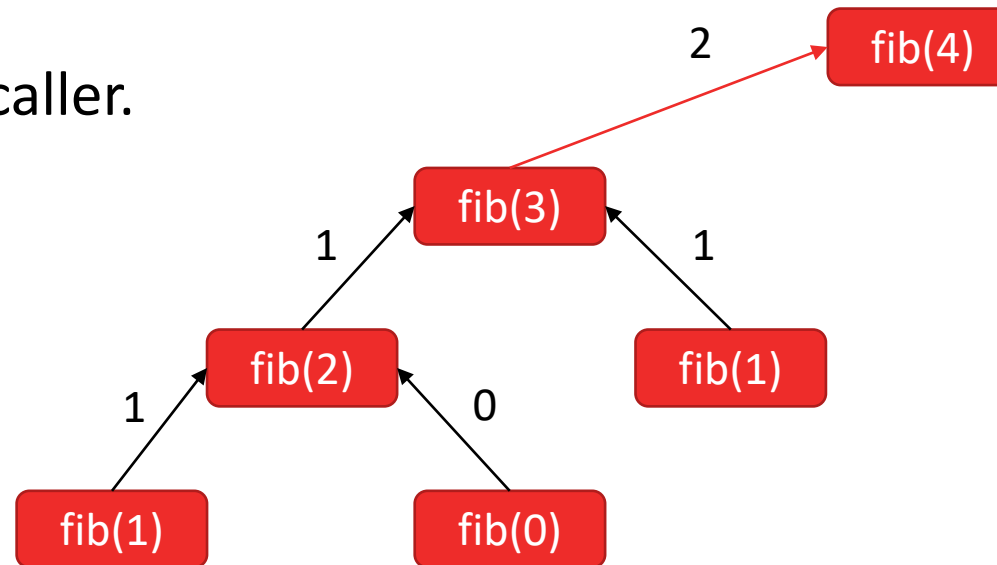


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

$fib(3) \rightarrow 1 + 1 = 2$

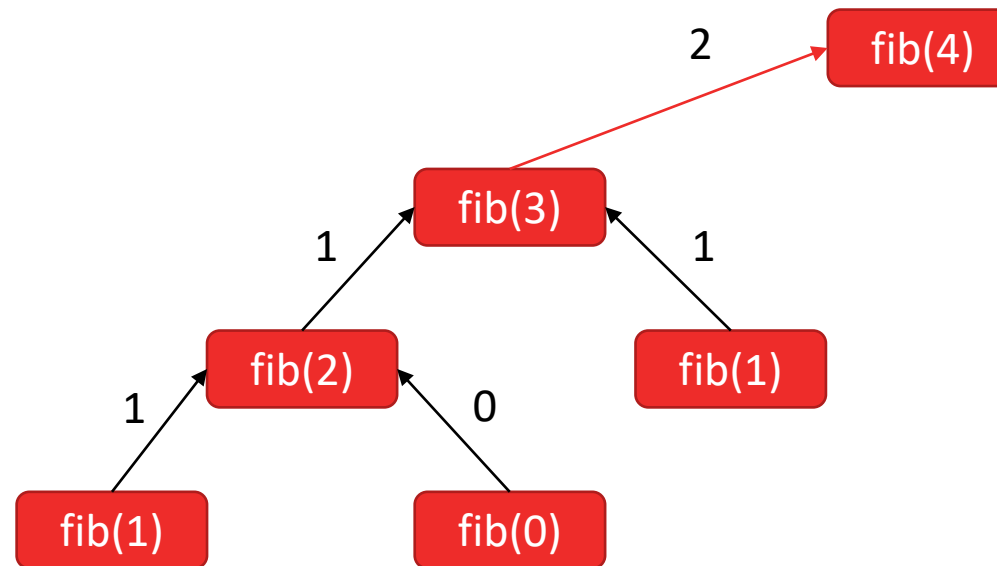
Now, $fib(3)$ returns 2 to the caller.



Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

$fib(4) \rightarrow 3 + fib(2)$

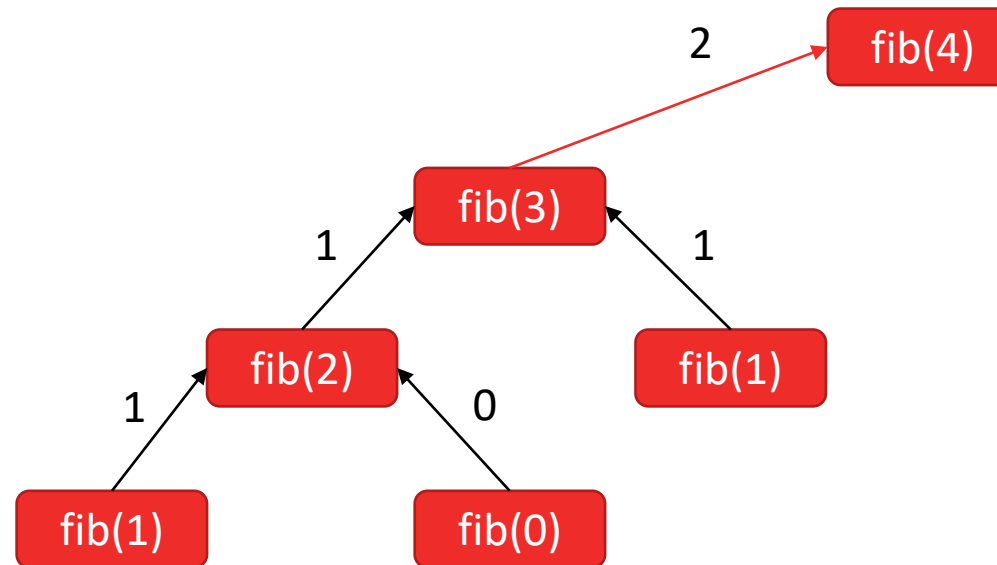


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

fib(4) → 3 + **fib(2)**

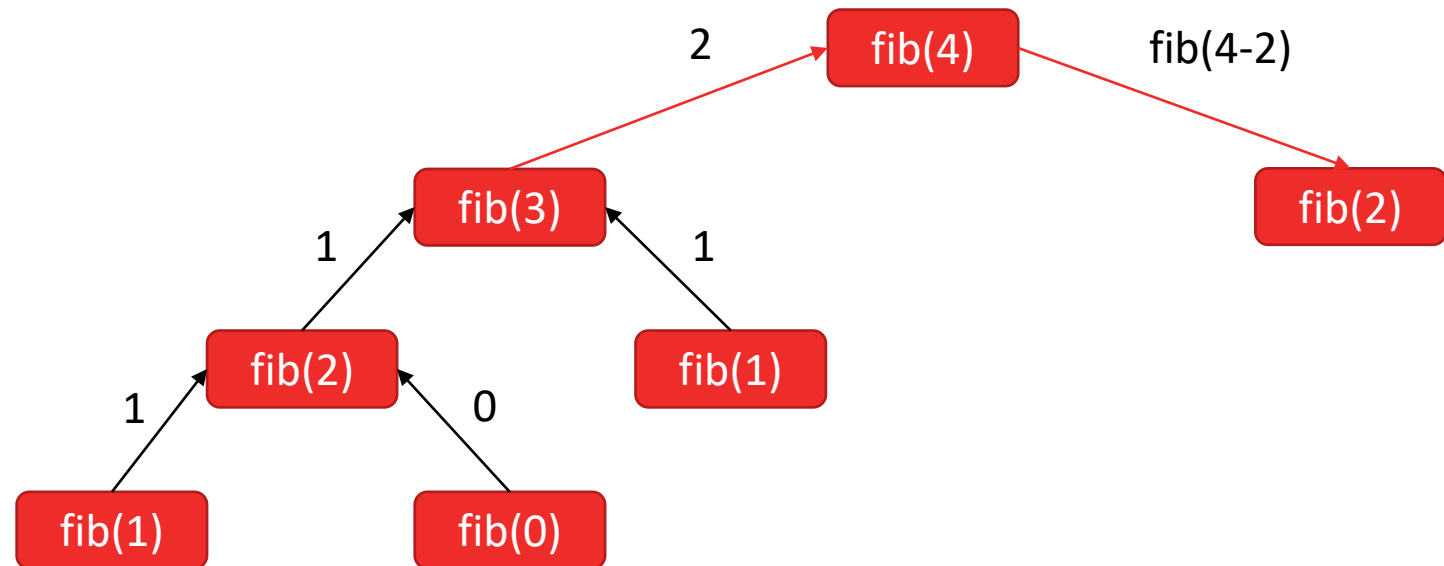
fib(2) is calculated.



Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

fib(2)

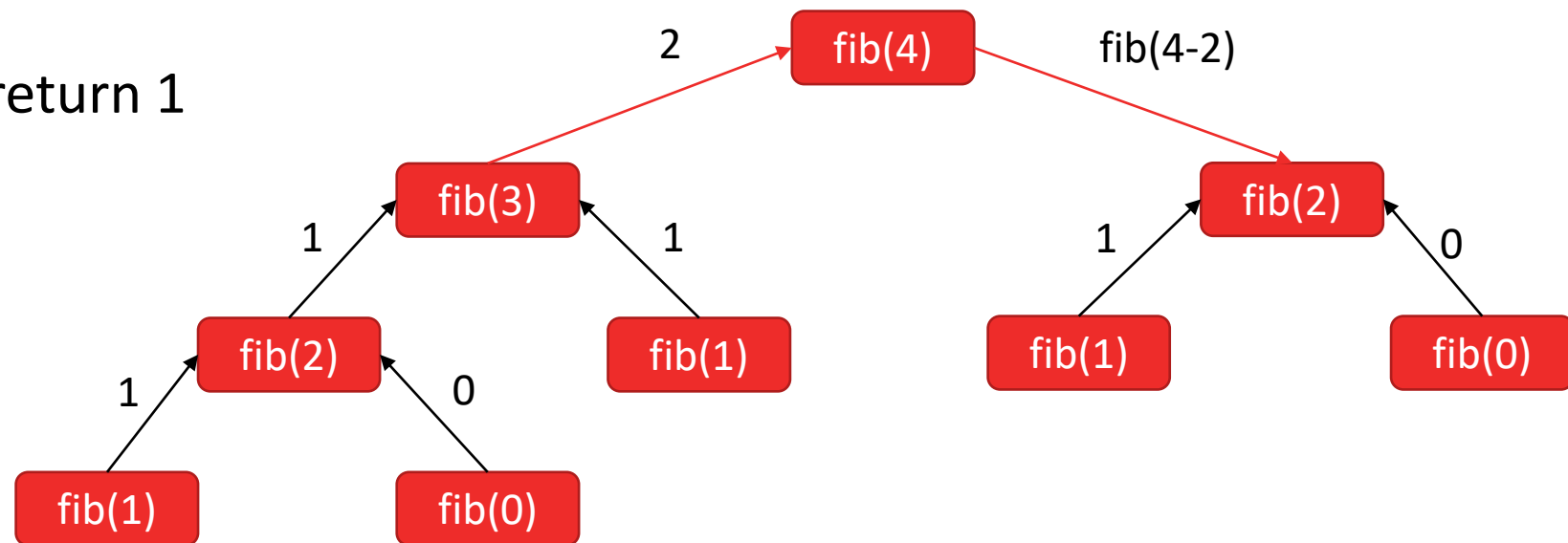


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{cases}$$

$fib(2) \rightarrow fib(1) + fib(0)$

Similar to before, $fib(2)$ will return 1

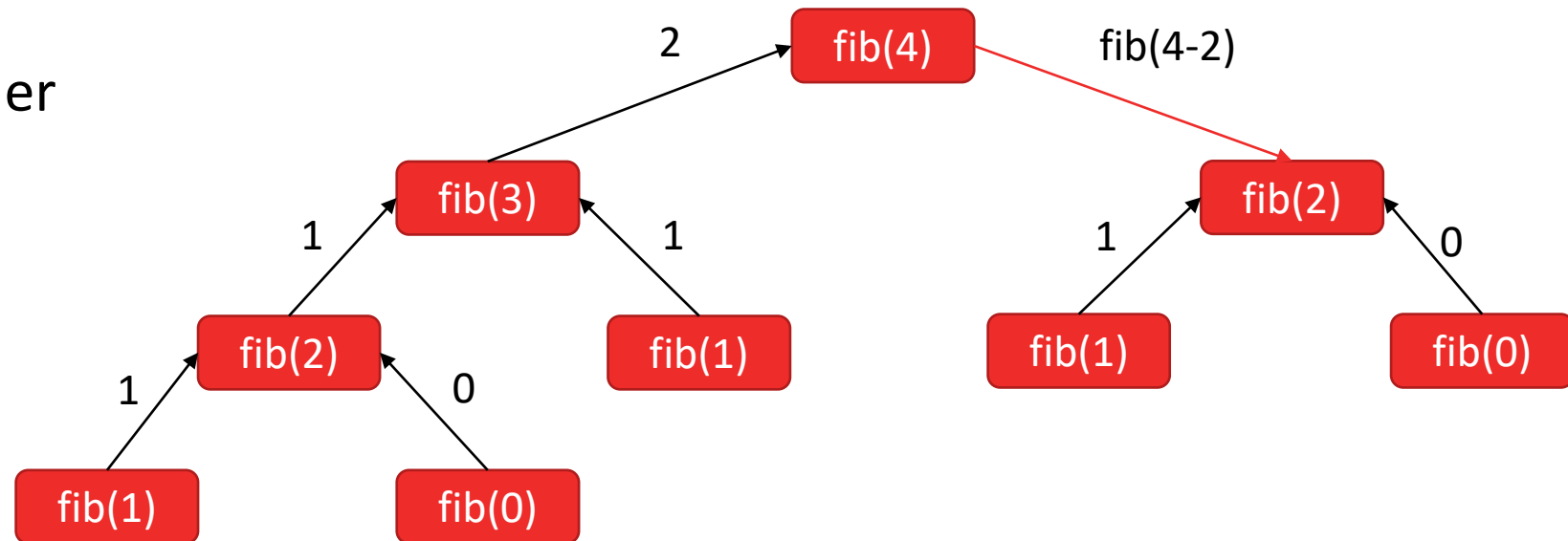


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

$fib(2) \rightarrow 1$

$fib(2)$ will return 1 to the caller

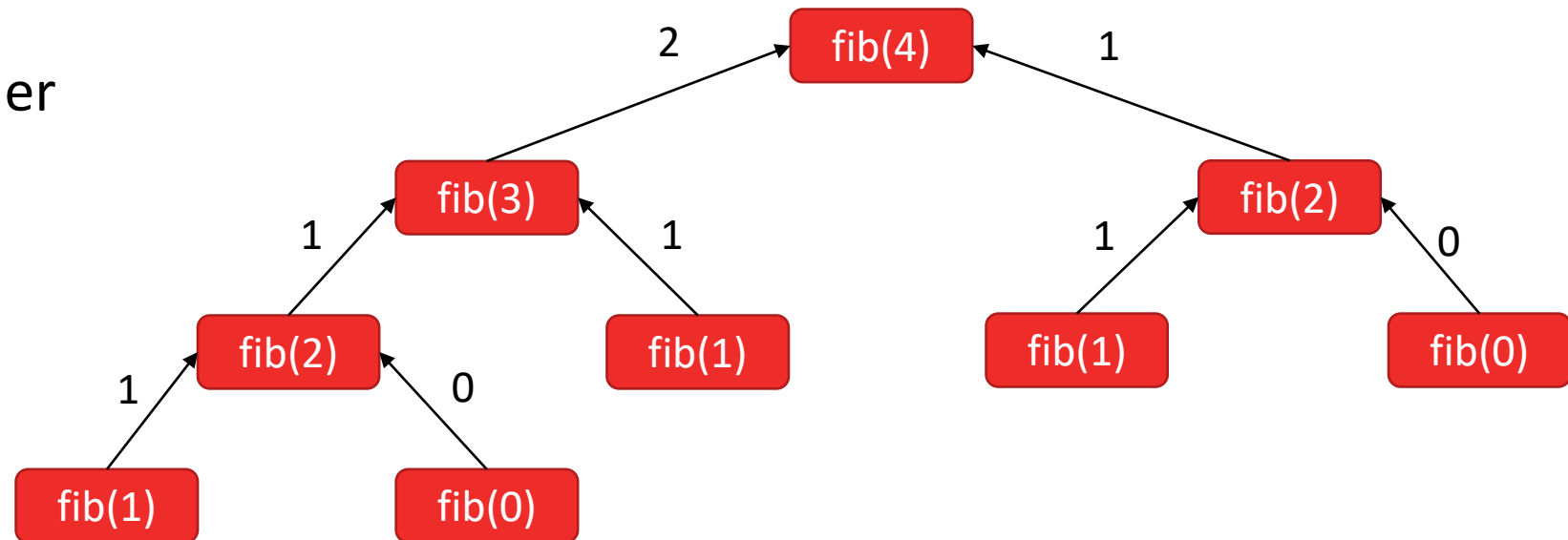


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

fib(2) → 1

fib(2) will return 1 to the caller

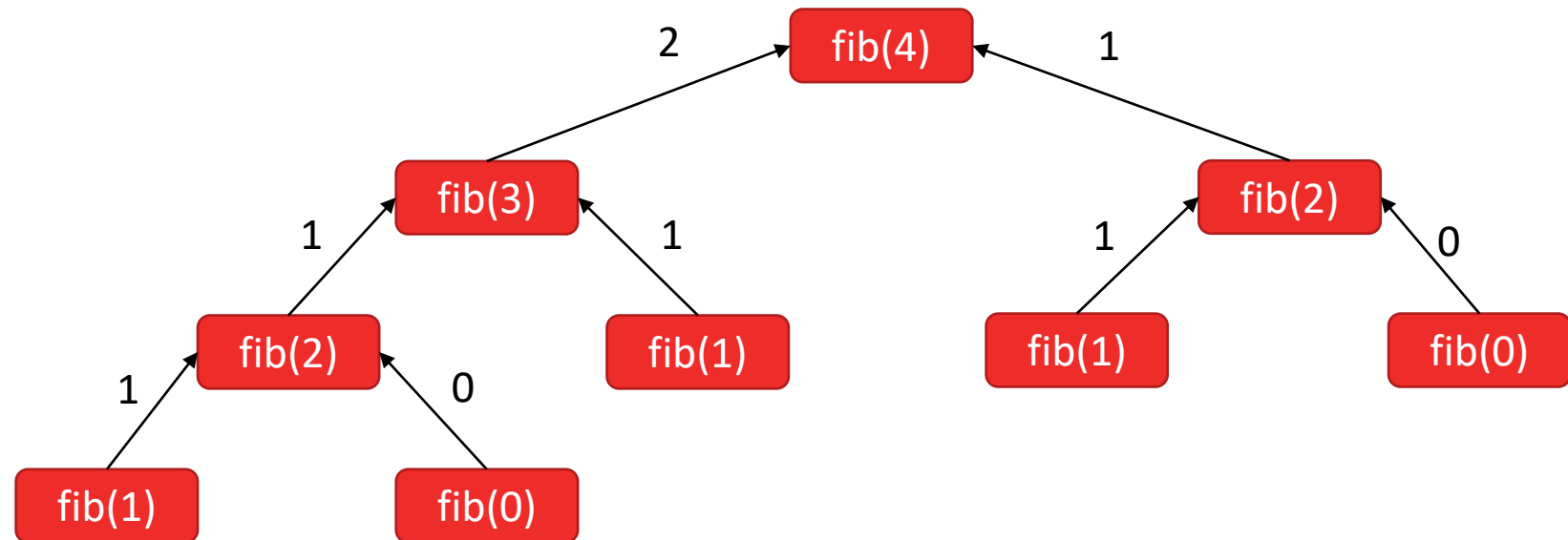


Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ fib(n - 1) + fib(n - 2) & n > 1 \end{cases}$$

$fib(4) \rightarrow 2 + 1 = 3$

Answer is 3.



Advantages

Advantages of Recursion

- Very well suited to some problems
 - Tree traversals
 - Flood fill
 - Fractal images
 - Quick sort / merge sort
 - ...
- Often easier to implement, sometimes faster than iterative

Notice Anything About Fibonacci Numbers

- ?

Notice Anything About Fibonacci Numbers

- We called fib(1), fib(0) many many many times
- Why not store this information once

Dynamic Programming

Dynamic Programming

n=	0	1	2	3	4	5	6	7	...
fib(n)	0	1	1	2	3	5	8	13	

```
def fibDyn(n):  
    fib = [0, 1]  
    for i in range(2, n+1):  
        fib.append(fib[i-2]+fib[i-1])  
    return fib[n]
```


Advantages of Dynamic Programming

- Dynamic programming is recursive in concept
 - but we store solutions we already found!
- We start bottom up instead of top down
 - Generally we find ourselves calculating many solutions at the bottom we find again and again
 - Let's calculate it once and store it
- Generally faster than similar recursive solution
 - We don't have to push information on the stack with recursion calls (we store our own state instead)

Advantages of Dynamic Programming

- Tricky to implement (especially for more complex problems see CPSC 413)
- But not always faster, or guaranteed to take less space than similar recursive solution
 - Sometimes we don't need need to track all this state, recursion lets us dispose of information stored on stack as the function call returns

Iterative Programming

Iterative Programming

- Sometimes we can notice in dynamic programming that we didn't need to store all the previous function calls
- Or we can noticed that each function call was base iteratively on the previous function calls

Iterative Programming

```
def fibIter(n):  
    if n == 0:  
        return 0  
    if n == 1:  
        return 1  
    prev1 = 1  
    prev2 = 0  
    result = None  
    for i in range(2, n+1):  
        result = prev1 + prev2  
        prev2 = prev1  
        prev1 = result  
    return result
```

Advantages of Iteration

- Typically
 - Faster (but not always)
 - Requires less memory (most of the time)
- But some problems are messy to express iteratively

Fractals

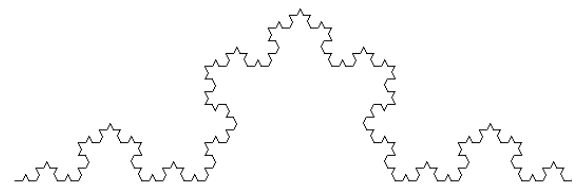
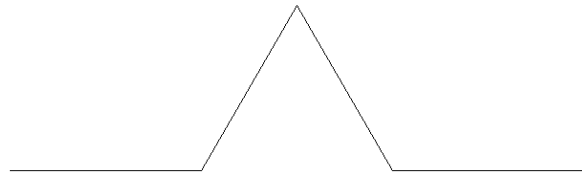
Fractals

- Self similar images
- Often have reasonably simple recursive definitions

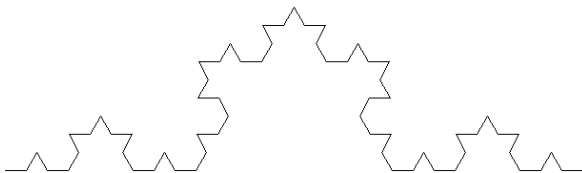
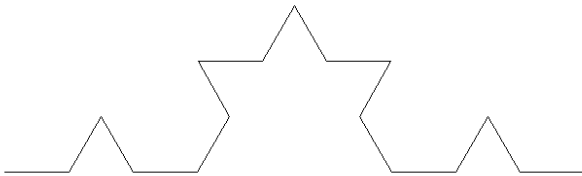
- 1

4

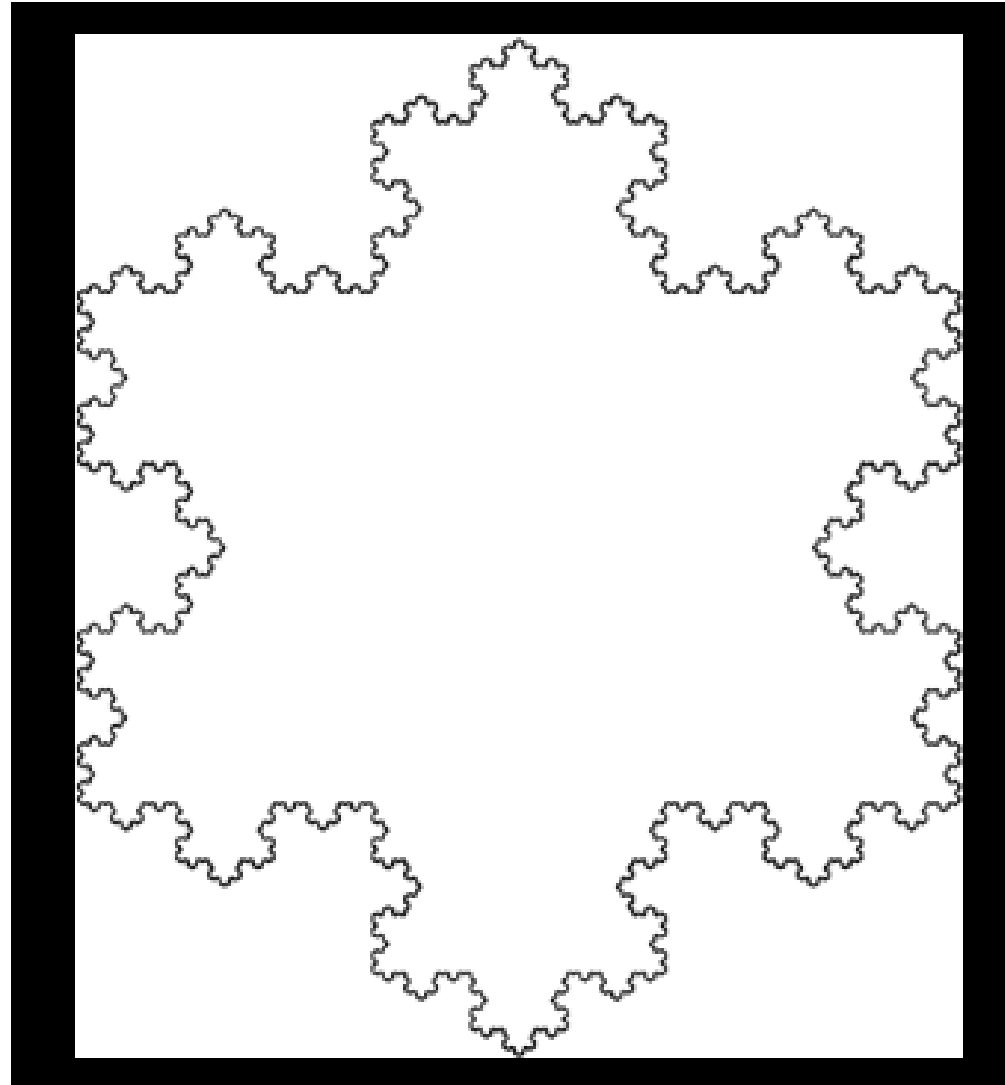
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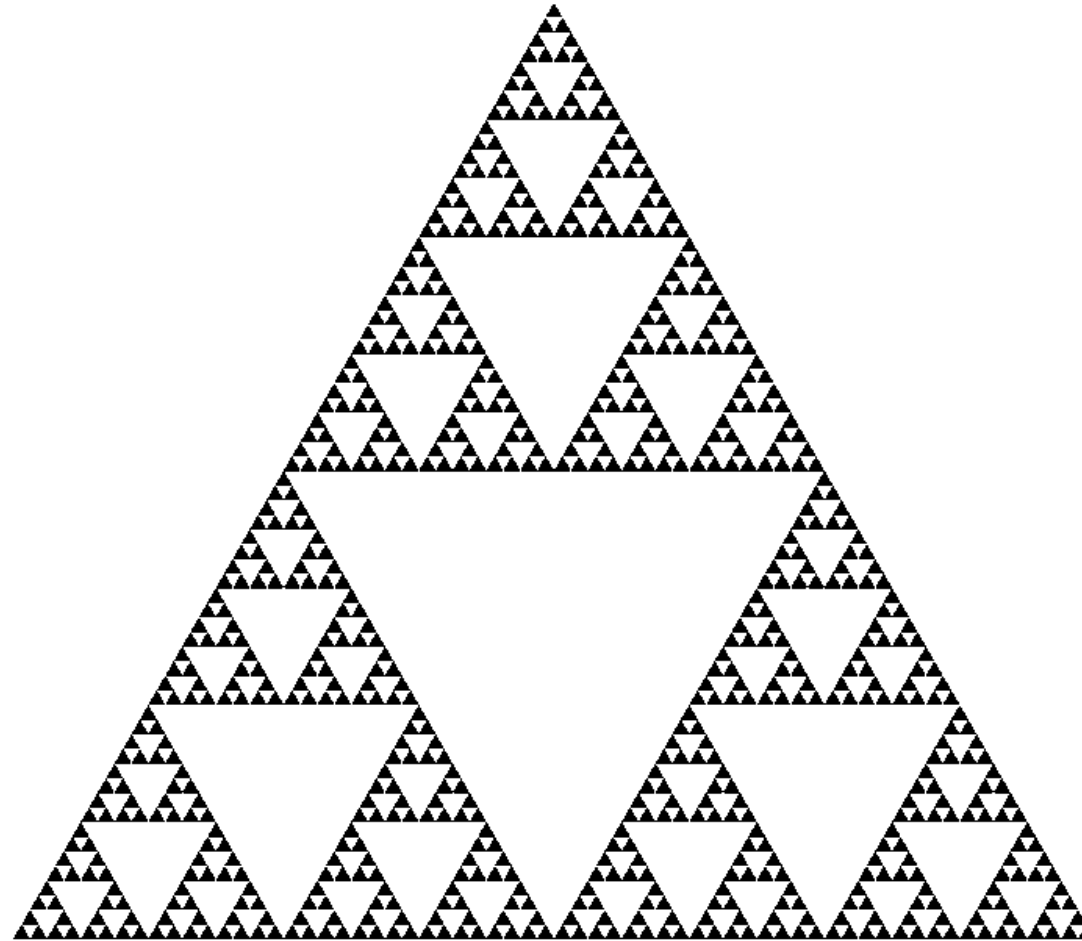
- 3



Koch Snowflake

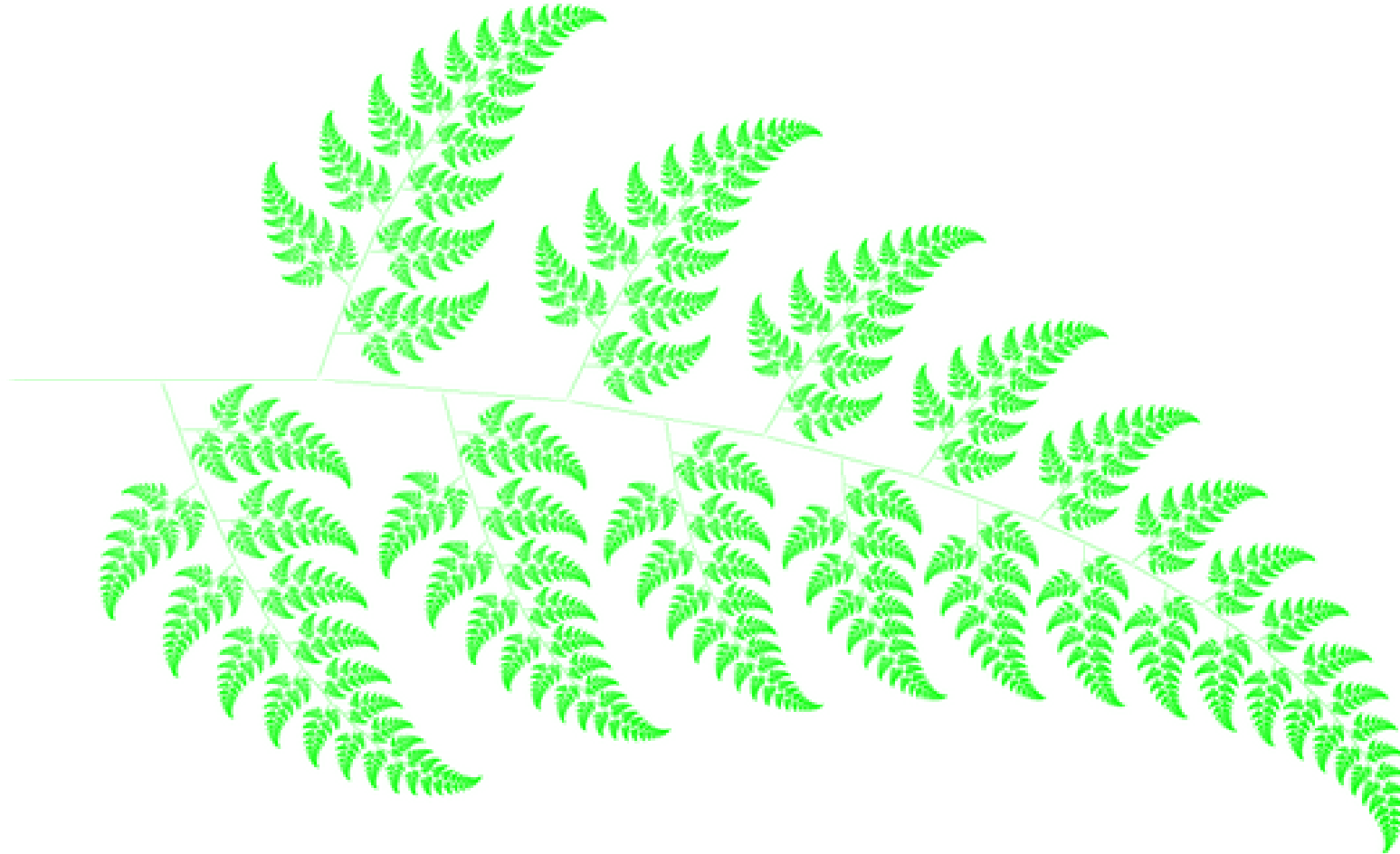


Sierpinski Triangle

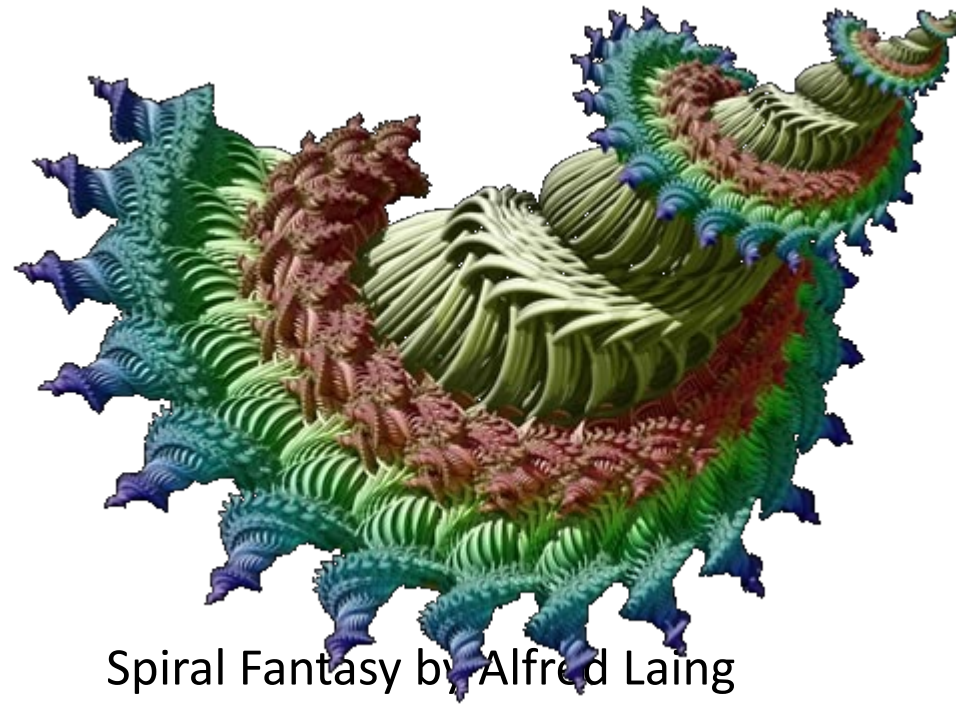


Sierpinski Trianglge
Source: <http://commons.wikimedia.org/wiki/File:Sierpinski-Trigon-7.svg>
License: Public Domain

Fractal Fern

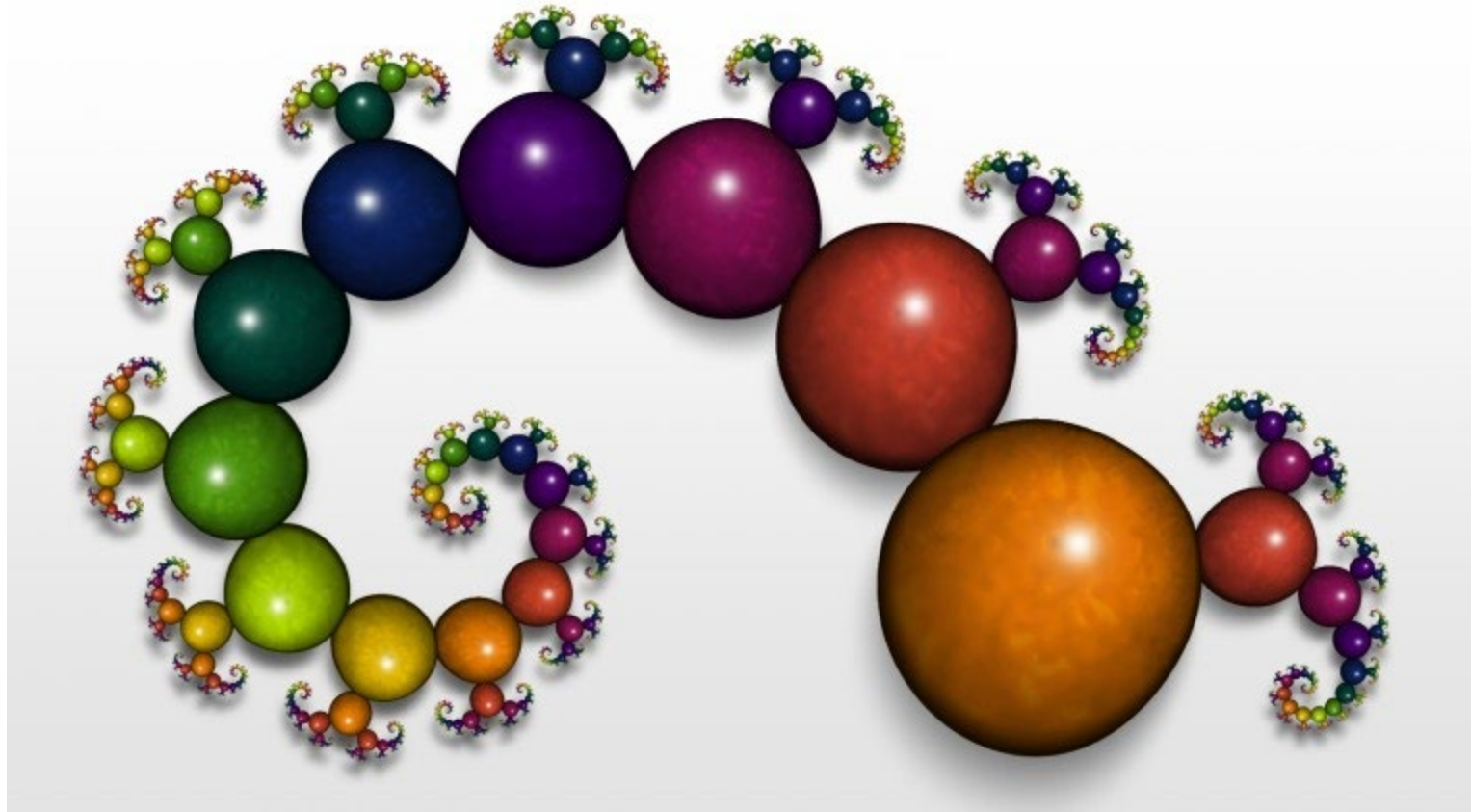


Fractal Art

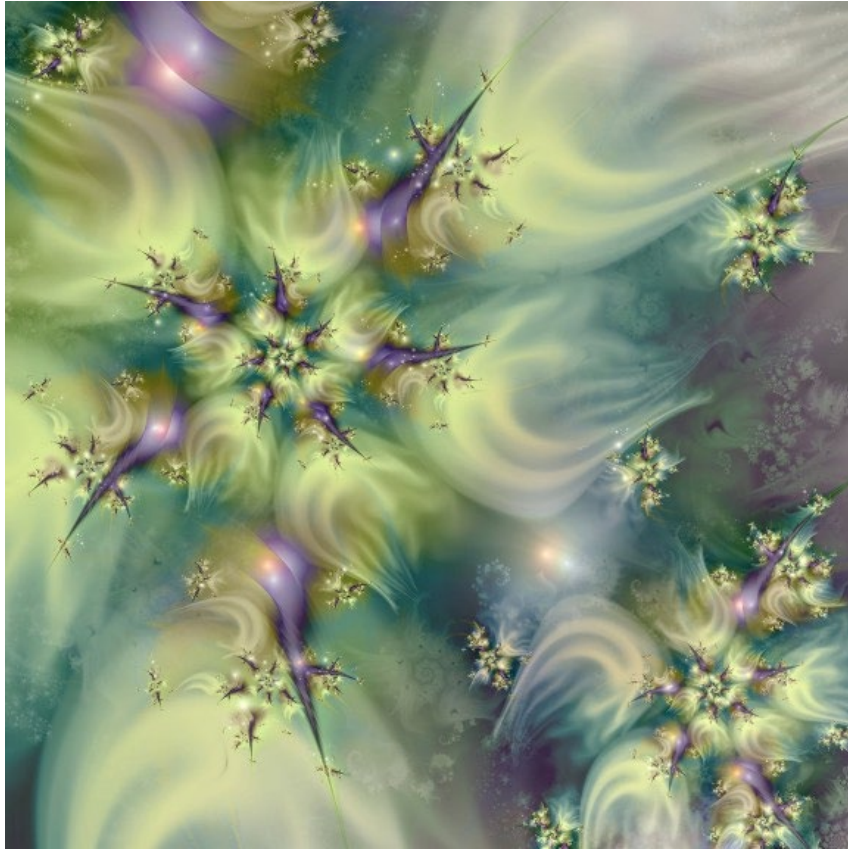


Spiral Fantasy by Alfred Laing

Fractal Art



Fractal Art



Rose by Keith Mack



Mazes

Maze Path Finding

- Consider a two dimensional list containing 4 different values
 - Entrance for the maze
 - Exit for the maze
 - Open spaces
 - Walls
- Assume that the maze is fully enclosed

Maze Path Finding

- Algorithm solve(map, x, y)
 - If the current square is a wall or a space we have already visited, return failure
 - If the current square is the exit point, mark it as part of the solution and return success
 - Mark the current square as part of the solution
 - If solve(map, x, y+1) is successful, return success
 - If solve(map, x, y-1) is successful, return success
 - If solve(map, x+1, y) is successful, return success
 - If solve(map, x-1, y) is successful, return success
 - Mark the current square as visited but not part of the solution
 - Return failure

Maze Path Finding

```
def solve(map, x, y):  
    if map[x][y] == "wall" or map[x][y] == "visited":  
        return False  
  
    if map[x][y] == "exit":  
        print "Go to (%d,%d)" % (x, y)  
        return True
```

Maze Path Finding

```
print "Go to (%d,%d)" % (x,y)
if solve(map,x,y+1):
    return True
elif solve(map,x,y-1):
    return True
elif solve(map,x+1,y):
    return True
elif solve(map,x-1,y):
    return True
else:
    map[x][y] = "visited"
    return False
```

Recursion – Beyond Algorithms

Not all about code

- We can also define structures that are recursive
- Most common examples are tree structures

```
class Node:
```

```
    def __init__(self):  
        self.children = [] #list of Node
```

```
root = Node()
```

```
root.children.append(Node())
```

```
child = root.children[0]
```

```
child.children.append(Node())
```

Binary Tree

- We can also define structures that are recursive
- Most common example is a binary tree

class Node:

```
def __init__(self, value):  
    self.value = None  
    self.left = None #Node with value less than self value  
    self.right = None #Node with value greater than self value
```

Heap

- We can also define structures that are recursive
- Already came up for Python's sort method called heapSort

class Node:

```
def __init__(self, value):  
    self.value = None  
    self.left = None #Node with value less than self value  
    self.right = None #Node with value less than self value
```

Recursion – Why?

Why recursion

- Recursive implementation is the perfect fit for naturally recursive problems.
- The result of one step depends on the previous steps.
- There is a base case where the process will stop at.
- e.g., search, games, various math problems
- Computationally elegant
- Easy to program than using loops
- Tradeoffs:
 - Function calls can be costly in terms of memory and time

Induction

- Recursive functions performance and even their results are often shown via proofs using induction
 - Base Case: Where recursion bottoms out and returns
 - Inductive Case: Where recursion calls another functions call
- Induction is generally be taught in CPSC 251, 331, and 413

Example

```
def binary_search(t_list, k):
    if len(t_list) != 0:
        center = len(t_list)//2
        if t_list[center] == k:
            return True
        if t_list[center] > k:
            return binary_search(t_list[:center], k)
        else:
            return binary_search(t_list[center+1:], k)
    return False
```

Onward to ... areas of Computer Science.

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<https://pages.cpsc.ucalgary.ca/~hudsonj/>



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CALGARY