Recursion

CPSC 217: Introduction to Computer Science for Multidisciplinary Studies I
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Recursion

- Definition:
  - See Recursion

  - Defining something in terms of itself
    - Generally using a smaller or simpler version

- Recursive Function
  - A function that calls itself

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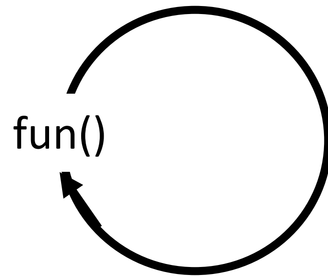
Recursion

- A programming technique whereby a function calls itself either directly or indirectly

```
def fun ():
    :
    fun ()
    :
```

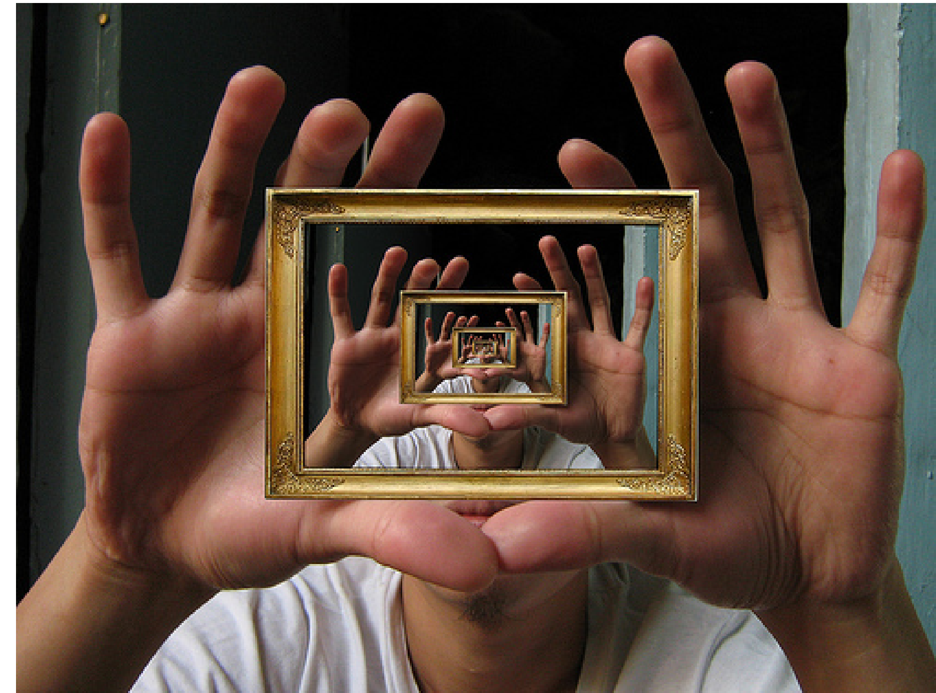


Directly

```
def fun1 ():
    fun2 ()

def fun2 ():
    fun1 ()
```

indirectly (mutual recursive)



Recursion

- A well-formed recursive function normally has two cases
  - Base Case:
    - Does not make a recursive call
    - Permits function to return (allowing us to fall out of the function calls)
    - Without a well-formed base case, you will have infinite recursive calls leading to the popular *StackOverFlowException* or equivalently *RecursionError* in Python
  - Recursive Case:
    - Function calls itself
    - Generally must be a call to a smaller or simpler version of the problem

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Tail Recursion

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Tail Recursion

- a tail call is a subroutine call performed as the final action of a function (Ex. A final return statement)
- If a tail call calls the function itself, then this function is called tail recursive

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Tail Recursion

- Typically when a function is called a record of memory (stack frame) is stored to track state from the previous function call
- But a tail recursive function doesn't need anything from before the final function call
- Modern compilers can identify this and reduce the process into an iterative loop for you. Although this is not guaranteed.
  - python is a procedural language, there is a class of languages called functional languages which will often guarantee this behaviour

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Usage of Recursion

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Useful Examples of Recursion

- Drawing fractals
- Finding a path through a maze
- Flood fill / “paint bucket” tool
- Merge sort, quick sort, binary search
- Finding the total size of all of the files in a directory and its subdirectories
- Parsing / evaluating expressions (how your code compiler knows you have syntax errors!)
- …

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Factorial

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A Small Example - Factorial

Computing n factorial (n!):

- Defined as the result of multiplying all numbers from n to 1 for all n>0. If n == 0, then result is 1.

$$n! = \begin{Bmatrix} 1 & n = 0 \\ n \times (n-1)! & n > 0 \end{Bmatrix}$$

- 0! → 1
- 1! → 1*1 = 1
- 2! → 2*1*1 = 2
- 3! → 3*2*1*1 = 6

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A Small Example - Factorial

Computing n factorial:

- Using a loop:
  - Initialize a variable named *answer* to 1 (the answer to 0!)
  - for i ranging from 2 to n (inclusive)
    - Multiply *answer* by i, storing the result back into *answer*
  - return *answer*

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A Small Example - Factorial

Computing n factorial:

- Using a loop:
  - Initialize a variable named *answer* to 1 (the answer to 0!)
  - for i ranging from 2 to n (inclusive)
    - Multiply *answer* by i, storing the result back into *answer*
  - return *answer*

- Another solution comes from the definition

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A Small Example

Recursive Definition!

- Another solution
  - By definition, $0! == 1$
  - View $n! == n * (n-1)!$

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A Small Example

Recursive Definition!

- Another solution
  - BASE CASE, 0! == 1
  - RECURSIVE CASE, n! == n * (n-1)!

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Factorial - Memory

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A Small Example - Factorial

```
def factorial(n):
    if n == 0: #BASE CASE
        return 1
    return n * factorial(n-1) #RECURSIVE CASE
```

Note: code is easy, directly follows from recursive definition

How does it look like in memory?

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A Small Example - Factorial

- Let’s trace this code:

```
def factorial(n):
    if n == 0: #BASE CASE
        return 1
    return n * factorial(n-1) #RECURSIVE CASE
factorial(3)
```


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A Small Example - Factorial - Walkthrough

Code:

```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

Trace:

STACK

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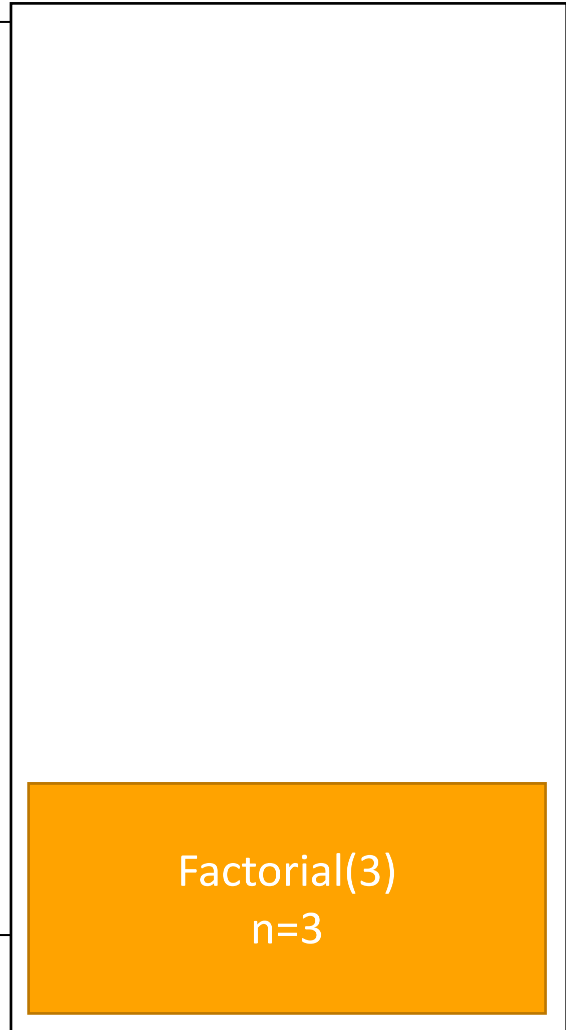
A Small Example - Factorial - Walkthrough

Code:

```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

Trace:

STACK



A Small Example - Factorial - Walkthrough

Code:

I1

```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

Trace:

```
n=3
```

STACK

Factorial(3)
n=3

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A Small Example - Factorial - Walkthrough

Code:

```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

I1 → `if n == 0:`

Trace:

```
n=3
```

STACK

Factorial(3) n=3

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A Small Example - Factorial - Walkthrough

Code:

FALSE


```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

Trace:

```
n=3
```

STACK

Factorial(3)
n=3

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A Small Example - Factorial - Walkthrough

Code:

```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

I1

Trace:

```
n=3
return 3*(factorial(2))
```

STACK

Factorial(3)
n=3

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A Small Example - Factorial - Walkthrough

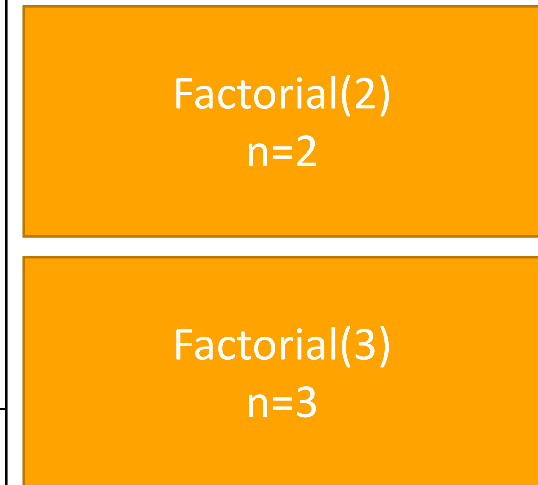
Code:

I2
```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```
I1

Trace:

```
n=2
return 3*(factorial(2))
```

STACK



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A Small Example - Factorial - Walkthrough

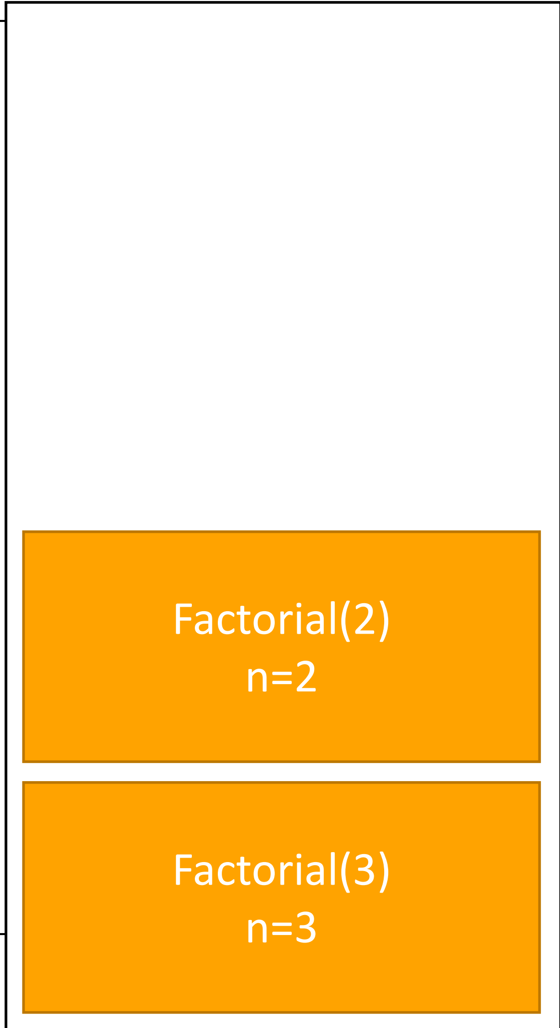
Code:

```
def factorial(n):
I2  if n == 0:
        return 1
I1  return n * factorial(n-1)
factorial(3)
```

Trace:

```
n=2
return 3*(factorial(2))
```

STACK



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A Small Example - Factorial - Walkthrough

Code:

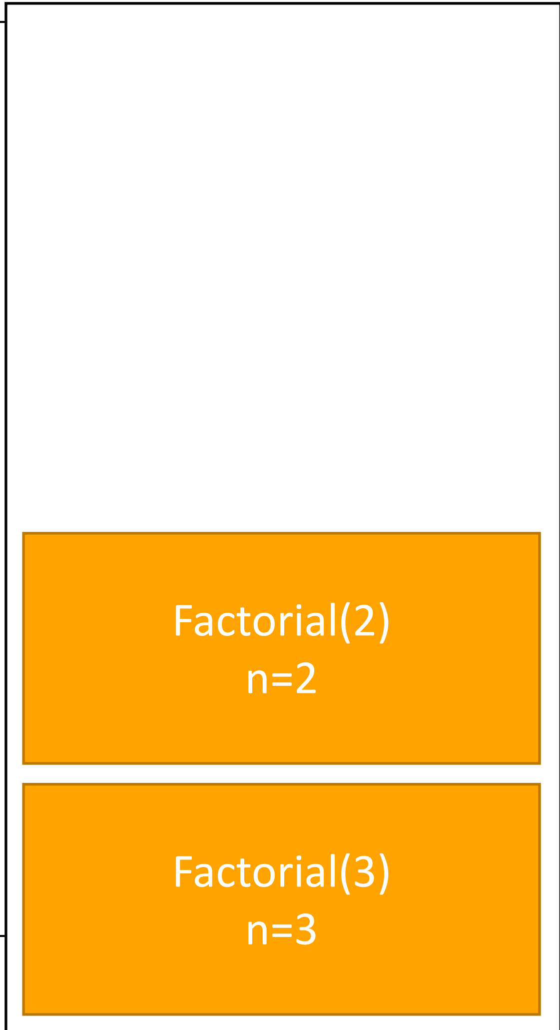
FALSE


```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

Trace:

```
n=2
return 3*(factorial(2))
```

STACK



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A Small Example - Factorial - Walkthrough

Code:

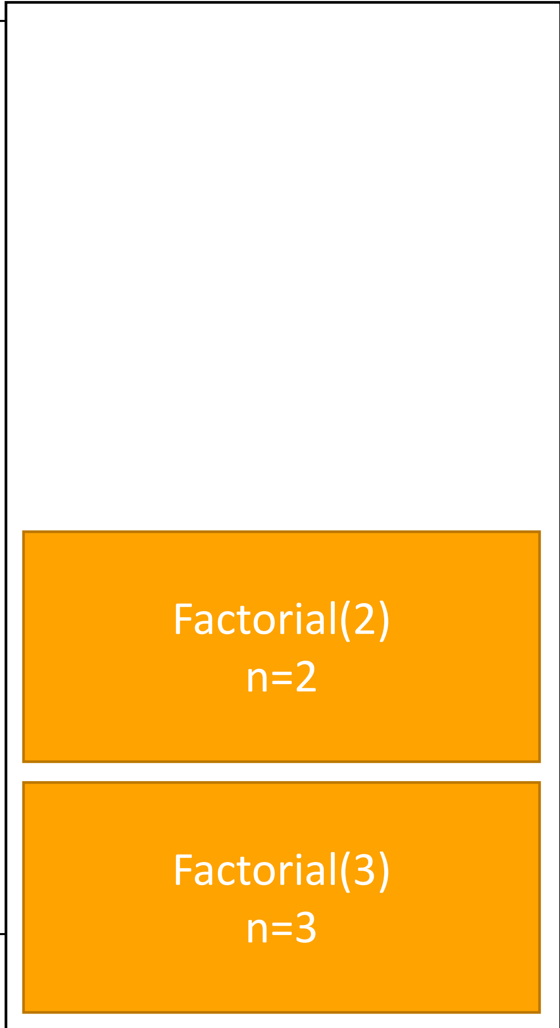
```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

I2

Trace:

```
n=2
return 3*(2*factorial(1))
```

STACK



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A Small Example - Factorial - Walkthrough

Code:

I3

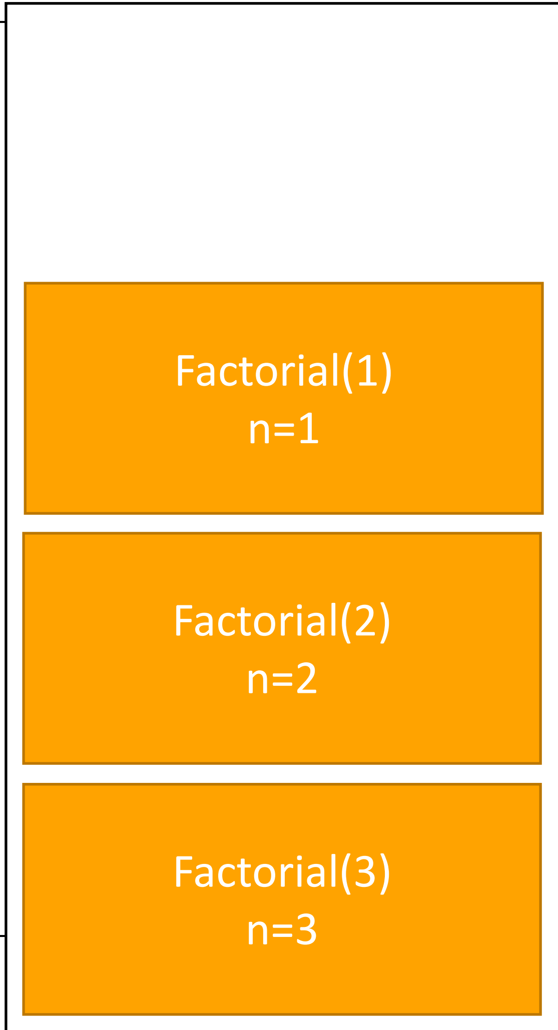
```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

I2

Trace:

```
n=1
return 3*(2*factorial(1))
```

STACK



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A Small Example - Factorial - Walkthrough

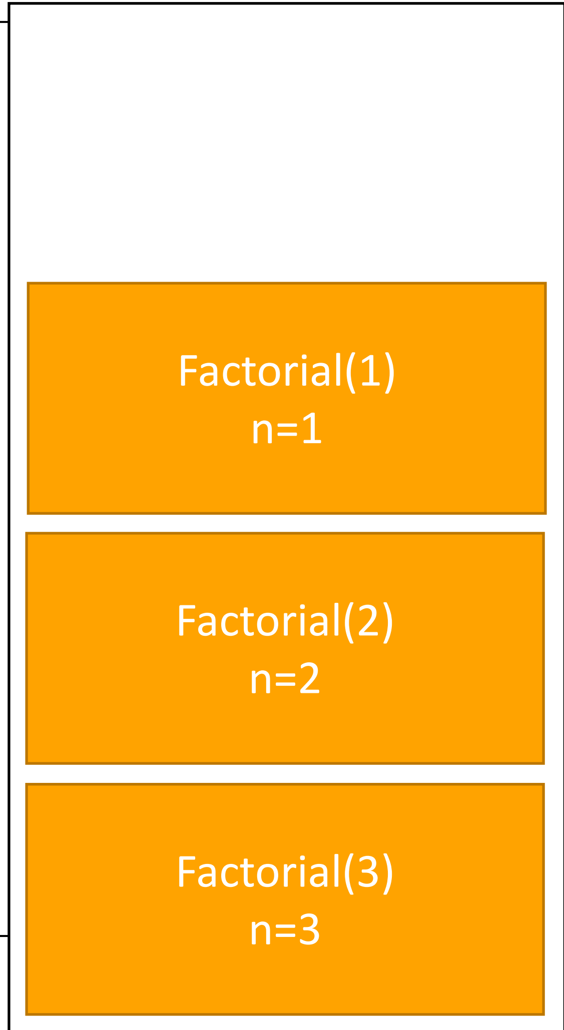
Code:

```
def factorial(n):
I3 → if n == 0:
        return 1
I2 → return n * factorial(n-1)
factorial(3)
```

Trace:

```
n=1
return 3*(2*factorial(1))
```

STACK



A Small Example - Factorial - Walkthrough

Code:

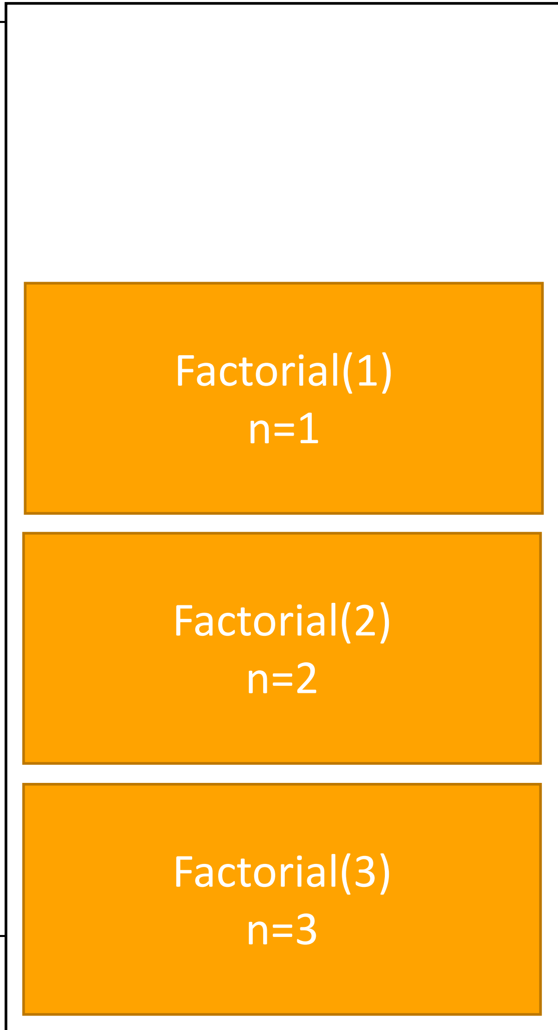
FALSE

```
def factorial(n):
I3→ if n == 0:
        return 1
I2→ return n * factorial(n-1)
factorial(3)
```

Trace:

```
n=1
return 3*(2*factorial(1))
```

STACK



A Small Example - Factorial - Walkthrough

Code:

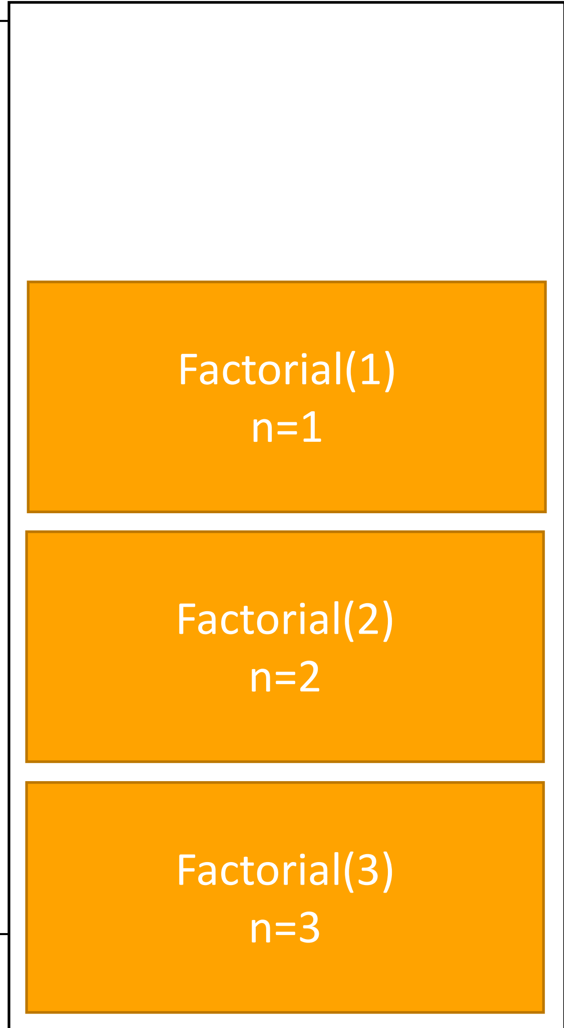
```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

I3

Trace:

```
n=1
return 3*(2*(1*factorial(0)))
```

STACK



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A Small Example - Factorial - Walkthrough

Code:

```
I4 ➜ def factorial(n):
         if n == 0:
             return 1
I3 ➜     return n * factorial(n-1)
     factorial(3)
```

Trace:

```
n=0
return 3*(2*(1*factorial(0)))
```

STACK

Factorial(1) n=0
Factorial(1) n=1
Factorial(2) n=2
Factorial(3) n=3

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A Small Example - Factorial - Walkthrough

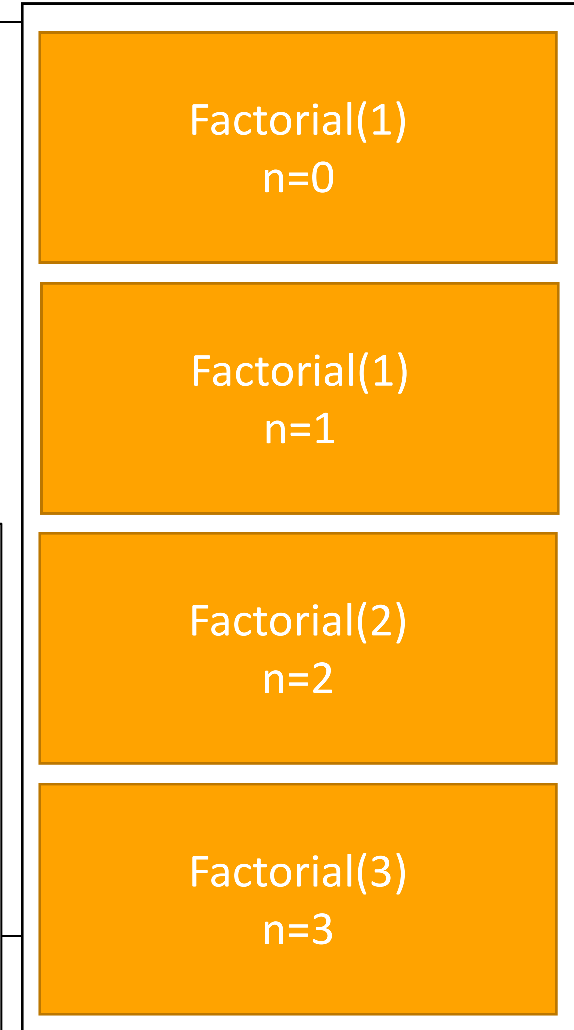
Code:

```
def factorial(n):
I4 →  if n == 0:
        return 1
I3 →  return n * factorial(n-1)
factorial(3)
```

Trace:

```
n=0
return 3*(2*(1*factorial(0)))
```

STACK



A Small Example - Factorial - Walkthrough

Code:

TRUE — I4 — I3

```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

Trace:

```
n=0
return 3*(2*(1*factorial(0)))
```

STACK

Factorial(1)
n=0

Factorial(1)
n=1

Factorial(2)
n=2

Factorial(3)
n=3

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A Small Example - Factorial - Walkthrough

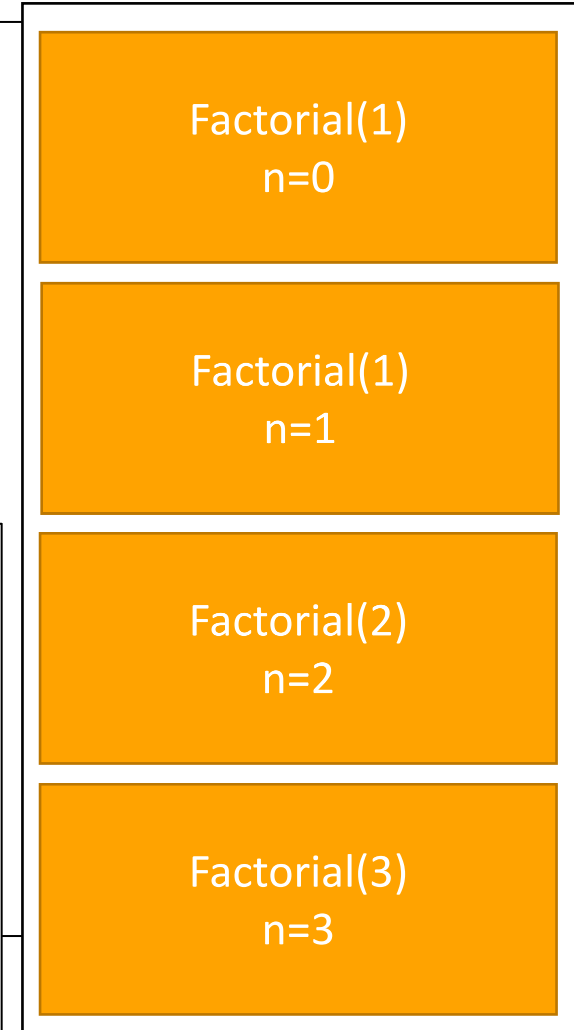
Code:

```
def factorial(n):
    if n == 0:
I4    return 1
I3  return n * factorial(n-1)
factorial(3)
```

Trace:

```
n=0
return 3*(2*(1*factorial(0)))
```

STACK



A Small Example - Factorial - Walkthrough

Code:

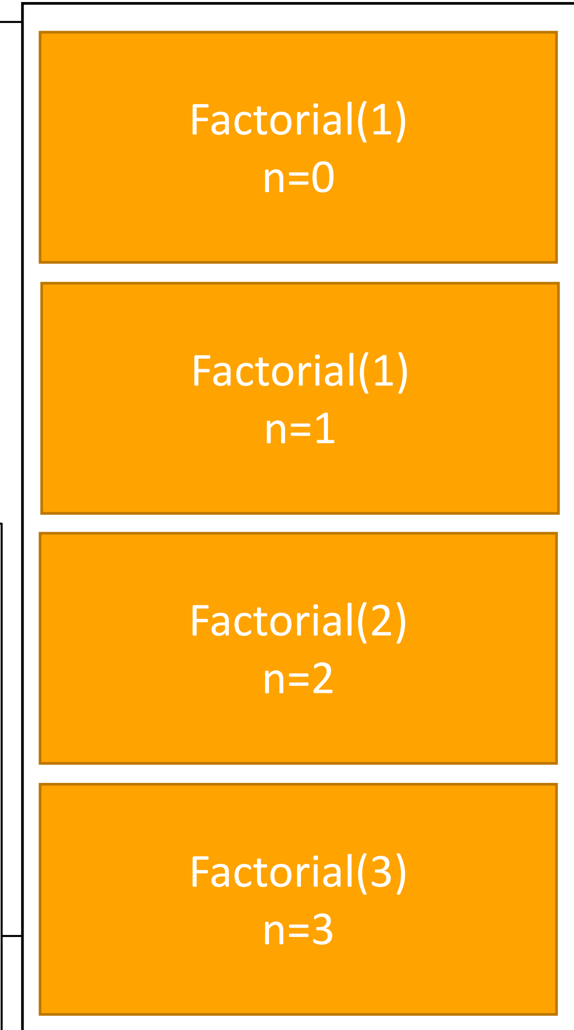
```
def factorial(n):
    if n == 0:
I4    return 1
I3  return n * factorial(n-1)
factorial(3)
```

Trace:

```
n=0
return 3*(2*(1*factorial(0)))
```

factorial(0) return 1

STACK



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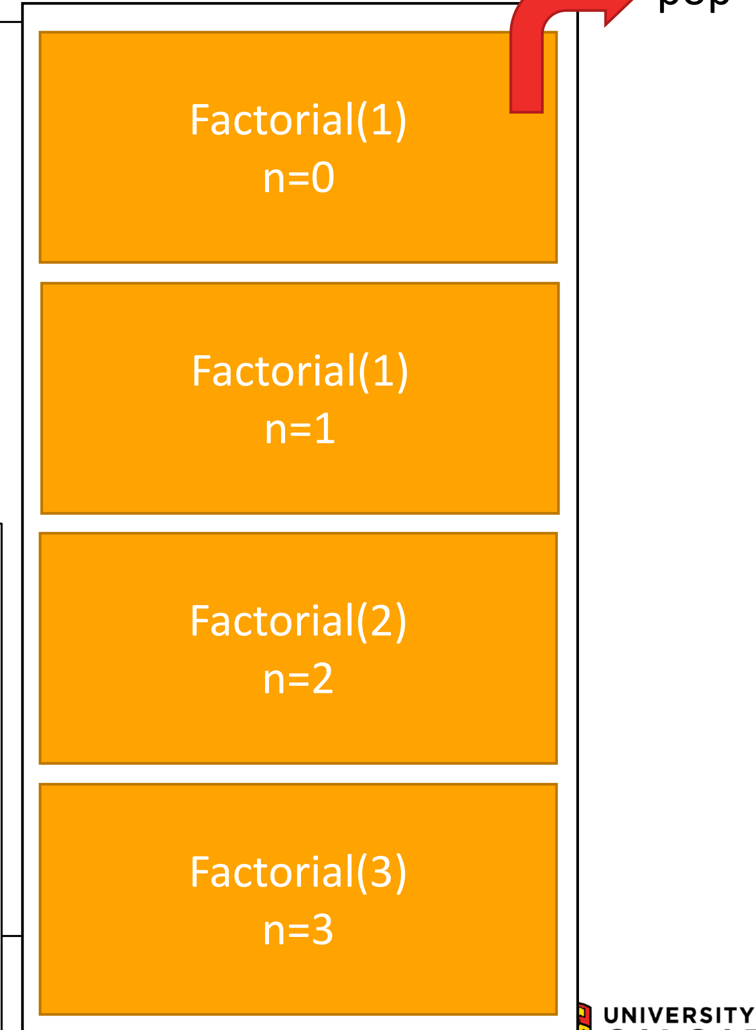
A Small Example - Factorial - Walkthrough

Code:

```
def factorial(n):
    if n == 0:
        return 1
I3  return n * factorial(n-1)
factorial(3)
```

Trace:

```
n=1
return 3*(2*(1*1))
```



A Small Example - Factorial - Walkthrough

Code:

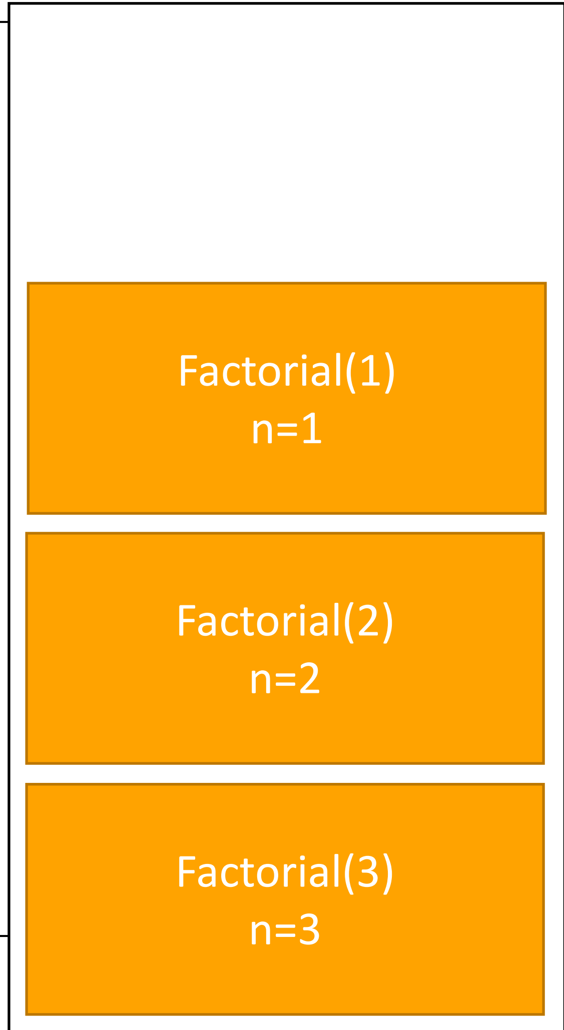
```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

I3

Trace:

```
n=1
return 3*(2*(1*1))
```

STACK



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A Small Example - Factorial - Walkthrough

Code:

```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

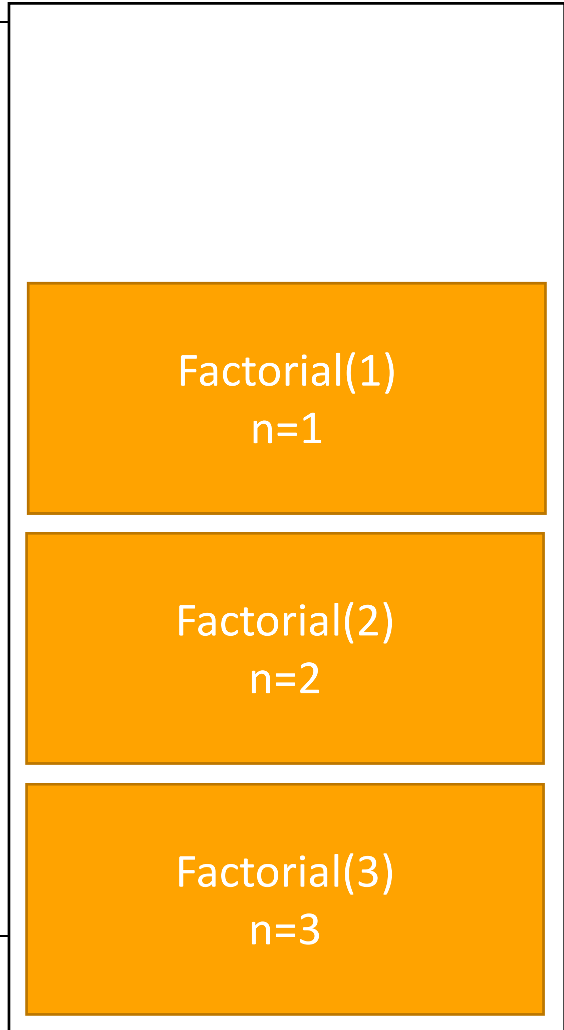
I3

Trace:

```
n=1
return 3*(2*(1*1))
```

factorial(1) returns
1*1 = 1

STACK



A Small Example - Factorial - Walkthrough

Code:

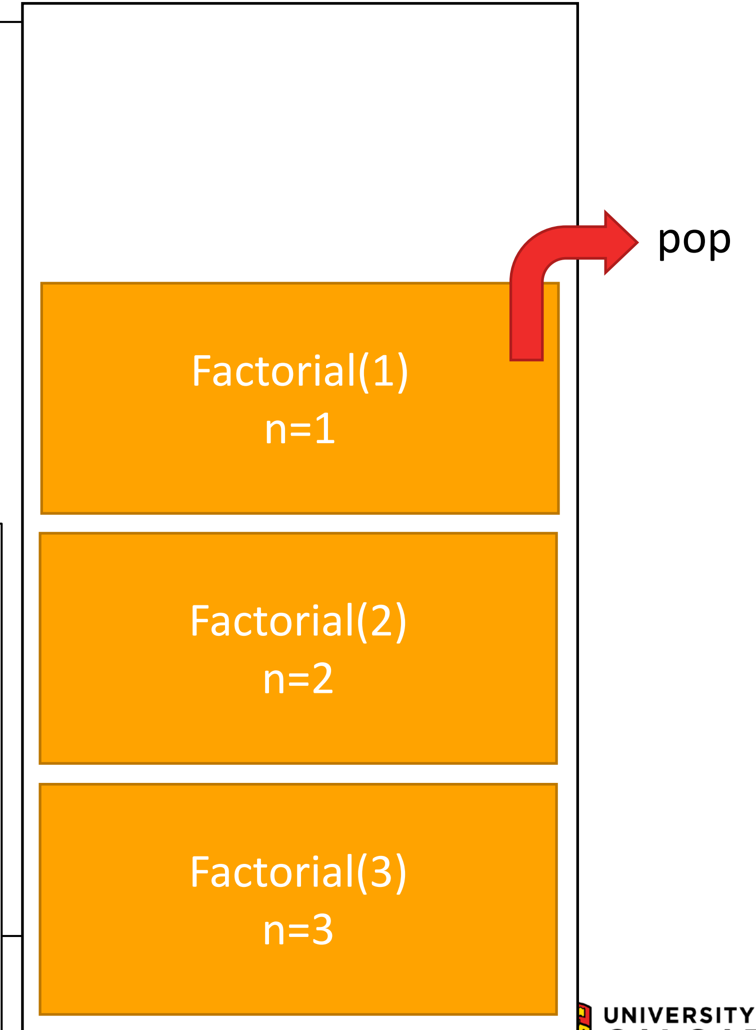
```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

I2

Trace:

```
n=2
return 3*(2*1)
```

STACK



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A Small Example - Factorial - Walkthrough

Code:

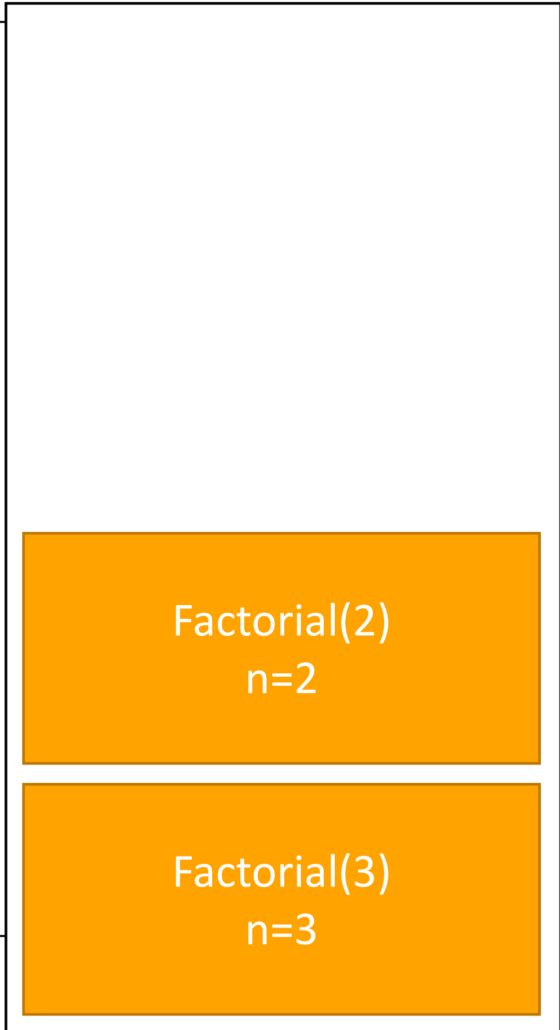
```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

I2

Trace:

```
n=2
return 3*(2*1)
```

STACK



A Small Example - Factorial - Walkthrough

Code:

```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

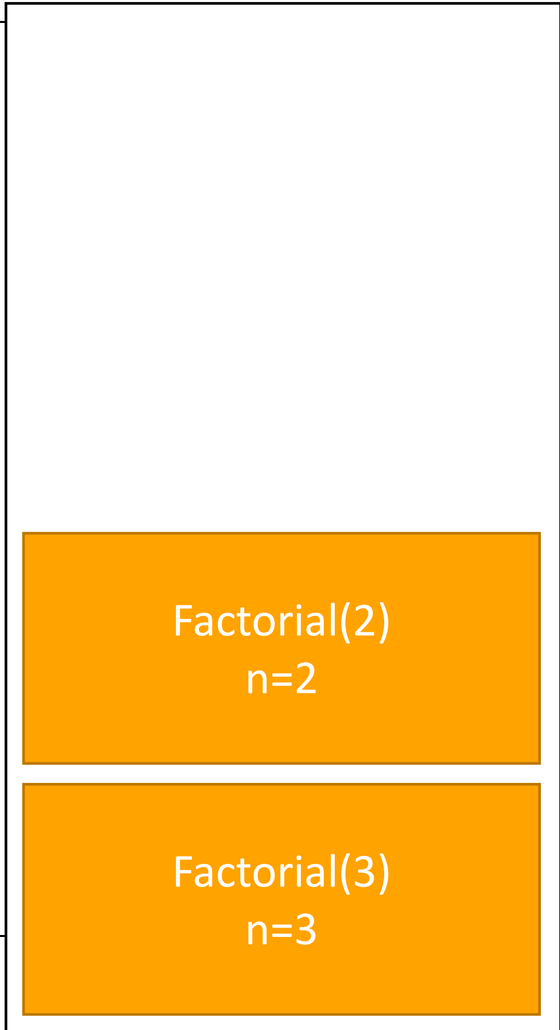
I2

Trace:

```
n=2
return 3*(2*1)
```

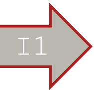
factorial(2) returns
2*1 = 2

STACK



A Small Example - Factorial - Walkthrough

Code:

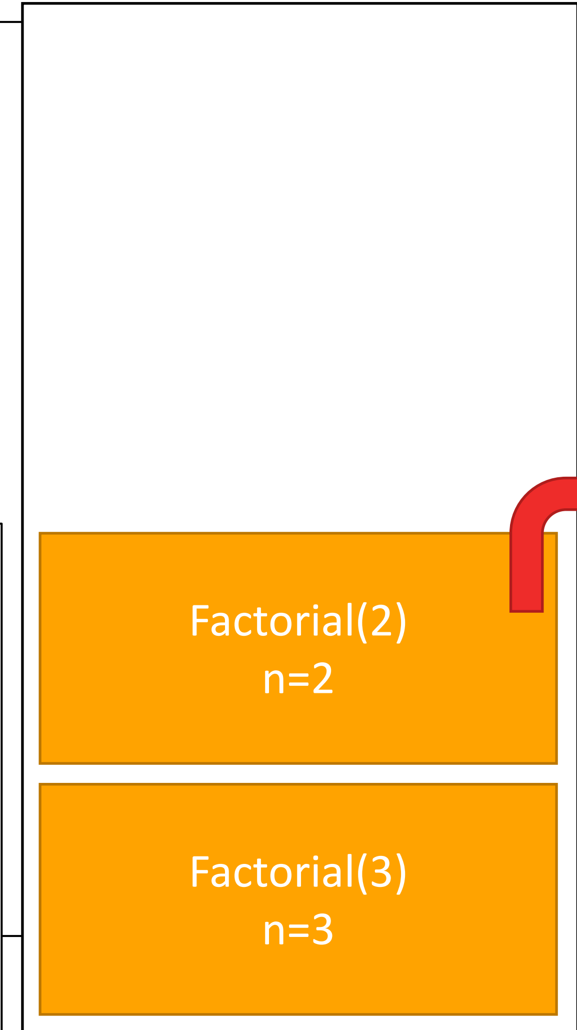



```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

Trace:

```
n=3
return 3*2
```

STACK



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A Small Example - Factorial - Walkthrough

Code:

```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```


Trace:

```
n=3
return 3*2
```

STACK

Factorial(3)
n=3

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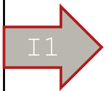
A Small Example - Factorial - Walkthrough

Code:

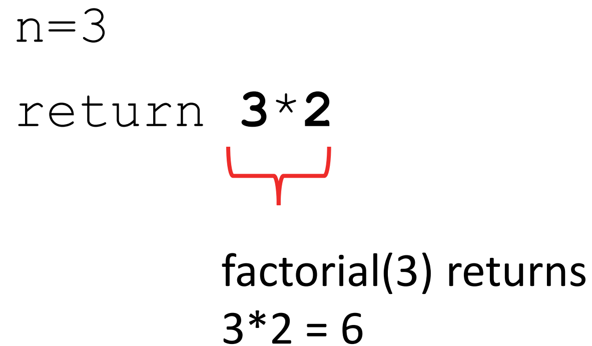
STACK

```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

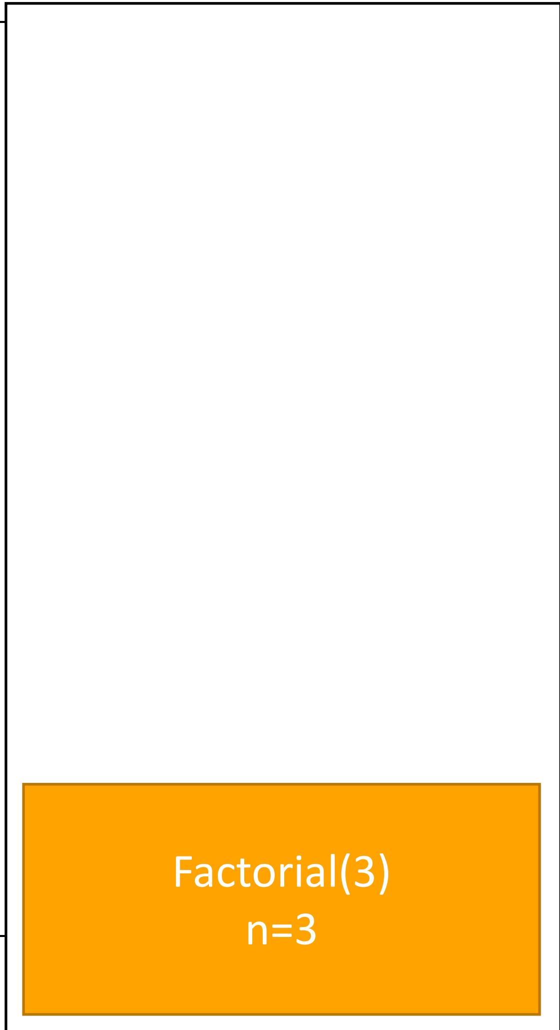


Trace:

```
n=3
return 3*2
```



factorial(3) returns
3*2 = 6



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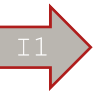
A Small Example - Factorial - Walkthrough

Code:

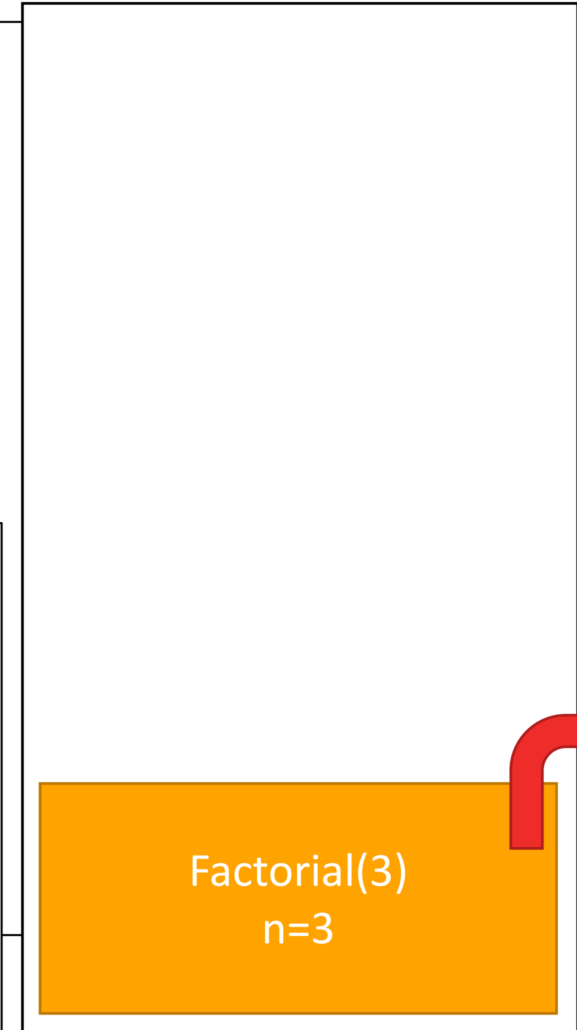
```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```



Trace:

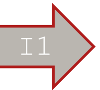
```
n=3
return 6
```

STACK



A Small Example - Factorial - Walkthrough

Code:




```
def factorial(n):
    if n == 0:
        return 1
    return n * factorial(n-1)
factorial(3)
```

Trace:

6

STACK

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Fibonacci

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Fibonacci Numbers

- A sequence of values:
  - 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

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Fibonacci Numbers

- A sequence of values:
  - 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …
- Defined recursively:
  - By definition:
    - fib(0) is 0
    - fib(1) is 1

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Fibonacci Numbers

- A sequence of values:
  - 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …
- Defined recursively:
  - By definition:
    - fib(0) is 0
    - fib(1) is 1
  - Remaining values:
    - Formed by computing the sum of the previous two values in the sequence

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Fibonacci Numbers

- A sequence of values:
  - 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …
- Defined recursively:
  - By definition:
    - fib(0) is 0
    - fib(1) is 1
  - Remaining values:
    - fib(n) = fib(n-1)+fib(n-2)

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Fibonacci Numbers

Note that there are two base cases here!

Eg. fib(4)

fib(4) = fib(3) + fib(2)

fib(3) = fib(2) + fib(1)

fib(2) = fib(1) + fib(0)

fib(1) = 1

fib(0) = 0

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Fibonacci Numbers

Note that there are two base cases here!

Eg. fib(4)

fib(4) = fib(3) + fib(2) =fib(1)+fib(0)+fib(1)+fib(1)+fib(0)=3

fib(3) = fib(2) + fib(1) = fib(1)+fib(0)+fib(1)=2

fib(2) = fib(1) + fib(0) =1

fib(1) = 1

fib(0) = 0

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Example - Fibonacci Numbers

Note that there are two base cases here!

Eg. fib(4)

fib(3)   fib(2)

fib(4) = fib(3) + fib(2) =fib(1)+fib(0)+fib(1) + fib(1)+fib(0)=3

fib(2)

fib(3) = fib(2) + fib(1) = fib(1)+fib(0)+fib(1)=2

fib(2) = fib(1) + fib(0) =1

fib(1) = 1

fib(0) = 0

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Fibonacci Numbers

Note that there are two base cases here!

Eg. fib(5)

fib(5) = fib(4) + fib(3) =fib(1)+fib(0)+fib(1)+fib(1)+fib(0)+ fib(1)+fib(0)+fib(1)=5

fib(4) = fib(3) + fib(2) =fib(1)+fib(0)+fib(1)+fib(1)+fib(0)=3

fib(3) = fib(2) + fib(1) = fib(1)+fib(0)+fib(1)=2

fib(2) = fib(1) + fib(0) =1

fib(1) = 1

fib(0) = 0

Example - Fibonacci Numbers

Note that there are two base cases here!

Eg. fib(5)

fib(4) fib(3)

fib(5) = fib(4) + fib(3) =fib(1)+fib(0)+fib(1)+fib(1)+fib(0)+ fib(1)+fib(0)+fib(1)=5

fib(4) = fib(3) + fib(2) =fib(1)+fib(0)+fib(1)+fib(1)+fib(0)=3

fib(3) = fib(2) + fib(1) = fib(1)+fib(0)+fib(1)=2

fib(2) = fib(1) + fib(0) =1

fib(1) = 1

fib(0) = 0

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Fibonacci Numbers

```
def fibRec(n):
    if n == 0:
        return 0
    if n == 1:
        return 1
    return fibRec(n-1) + fibRec(n-2)
```


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Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(4)

fib(4)

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Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(4) → fib(3) + fib(2)

fib(4)

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Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(4) → **fib(3)** + fib(2)

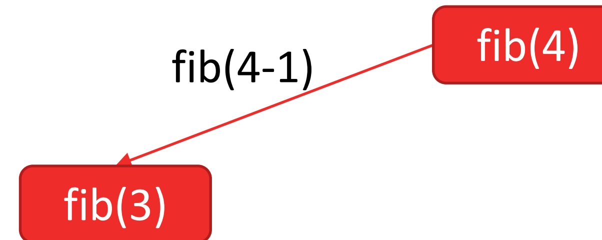
The expression is evaluated left to right.
So, fib(3) is calculated first.

fib(4)

Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(3)



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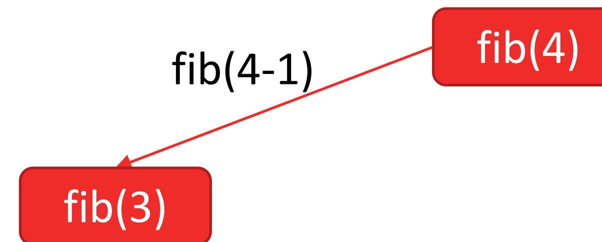
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(3) → fib(2) + fib(1)



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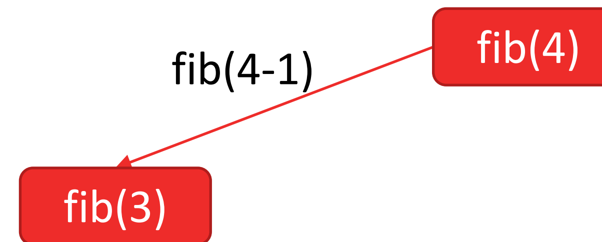
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(3) → **fib(2)** + fib(1)

Similarly, fib(2) is evaluated first.



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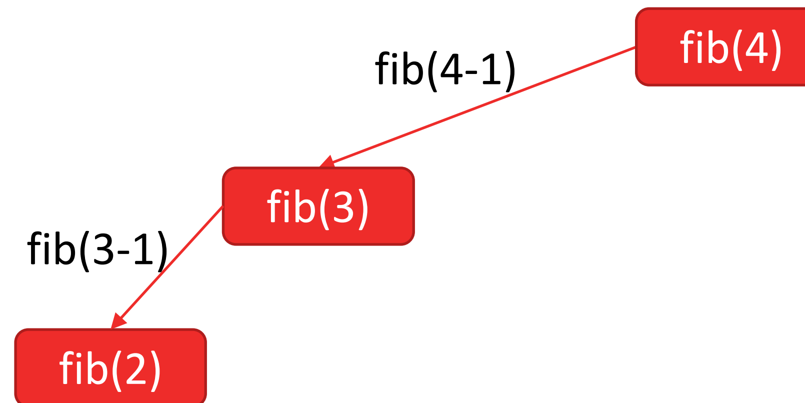
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(2)



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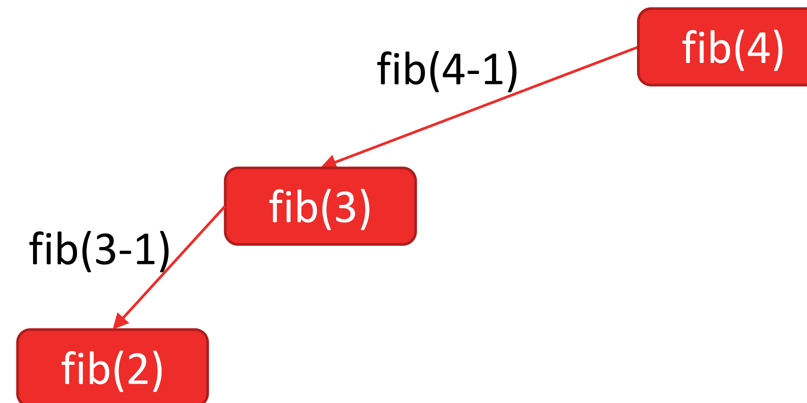
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(2) → fib(1) + fib(0)



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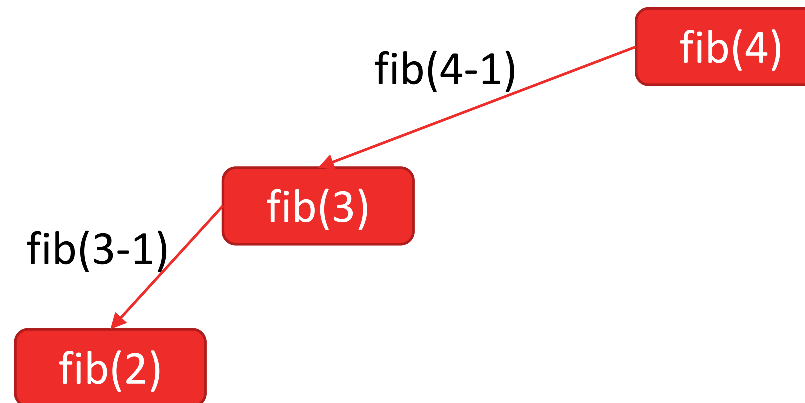
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(2) → **fib(1)** + fib(0)

fib(1) is evaluated first



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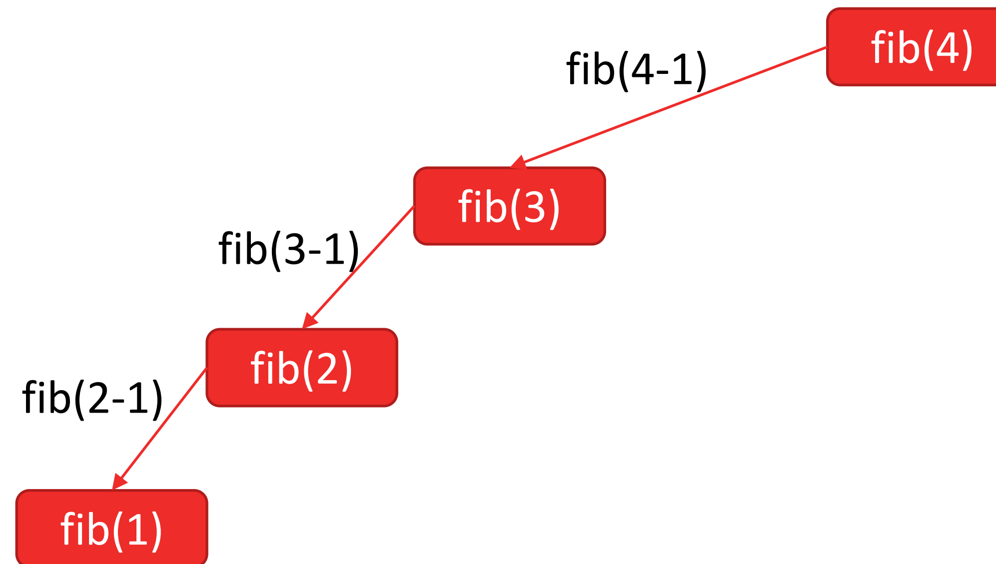
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(1)



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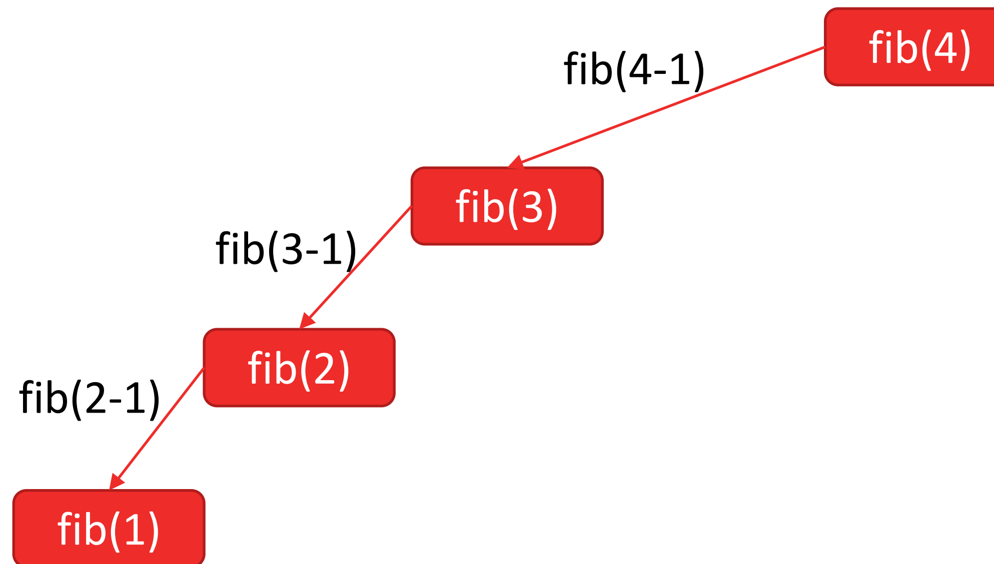
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(1) → 1

fib(1) returns 1 to caller



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Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

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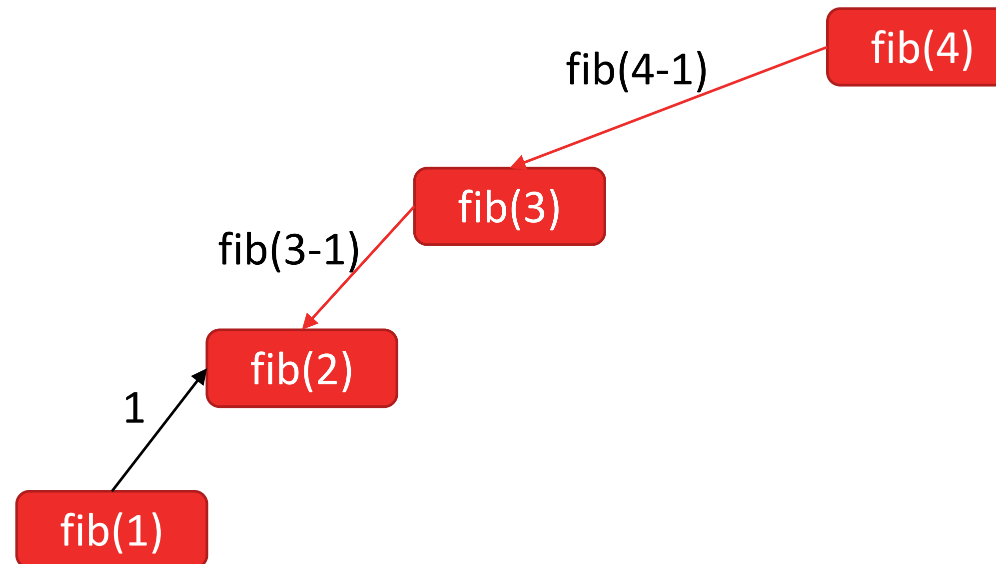

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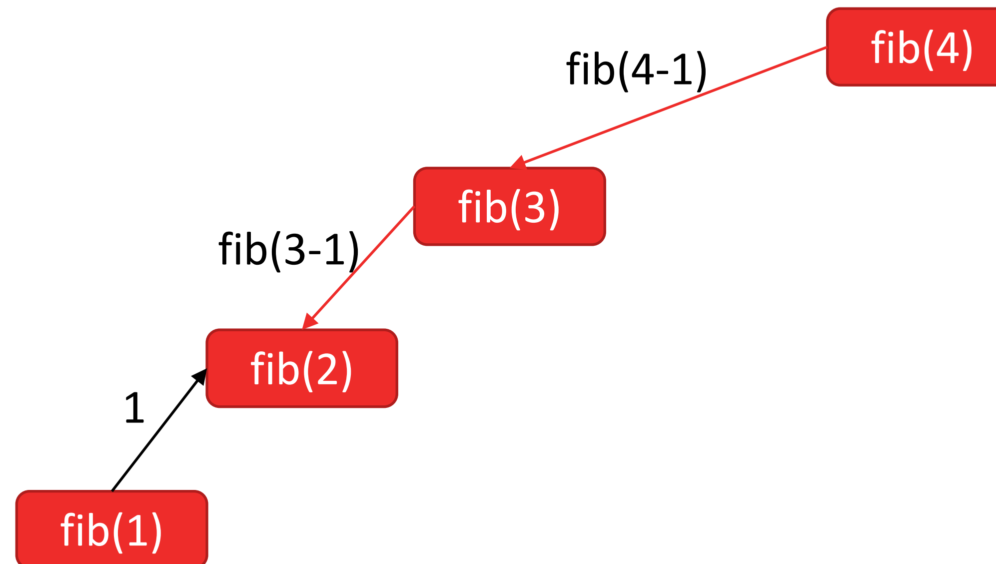
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(2) → 1 + **fib(0)**

Now, fib(0) is evaluated.



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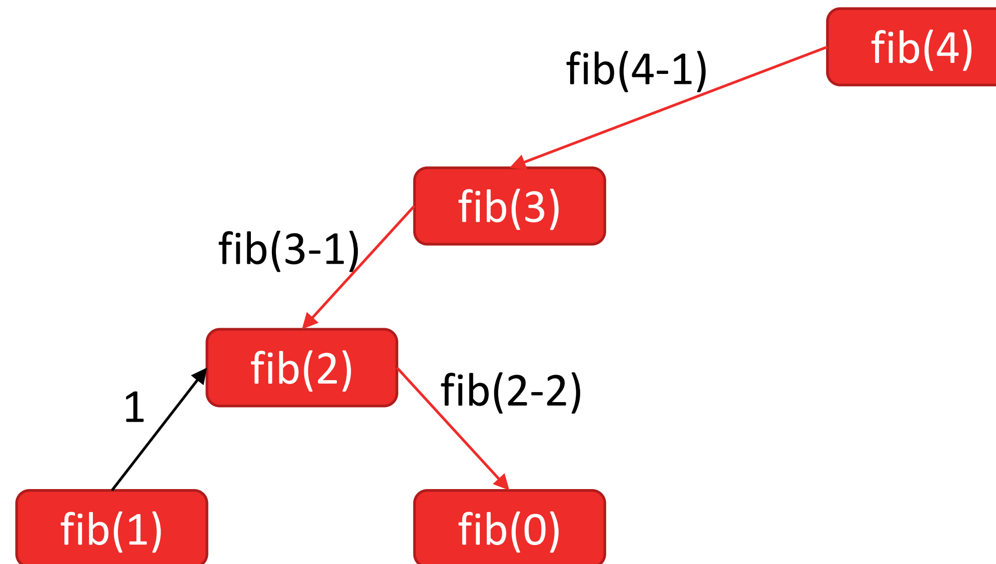
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(0)



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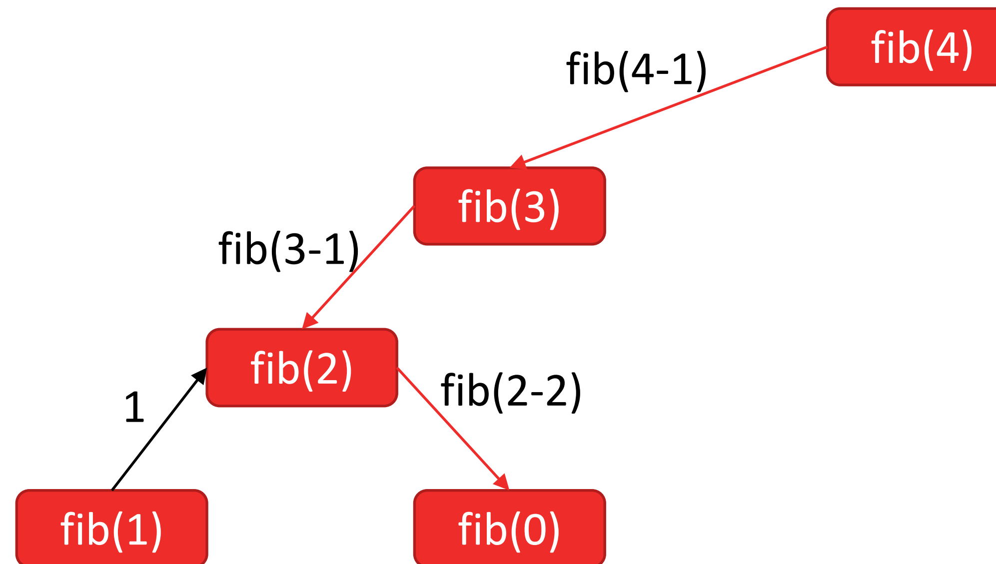
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(0) → 0

fib(0) returns 0 to caller



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Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

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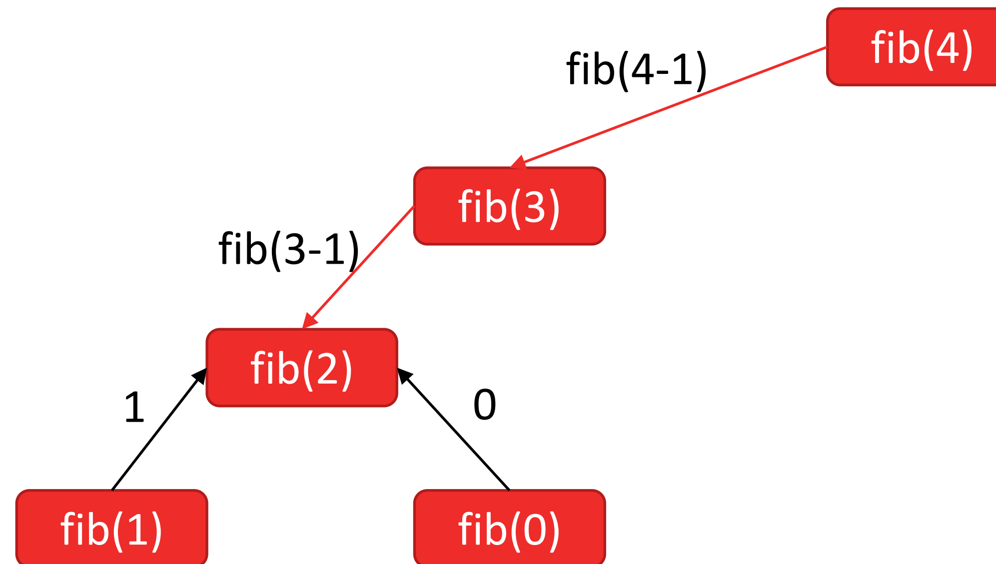
fib(0) returns 0 to caller



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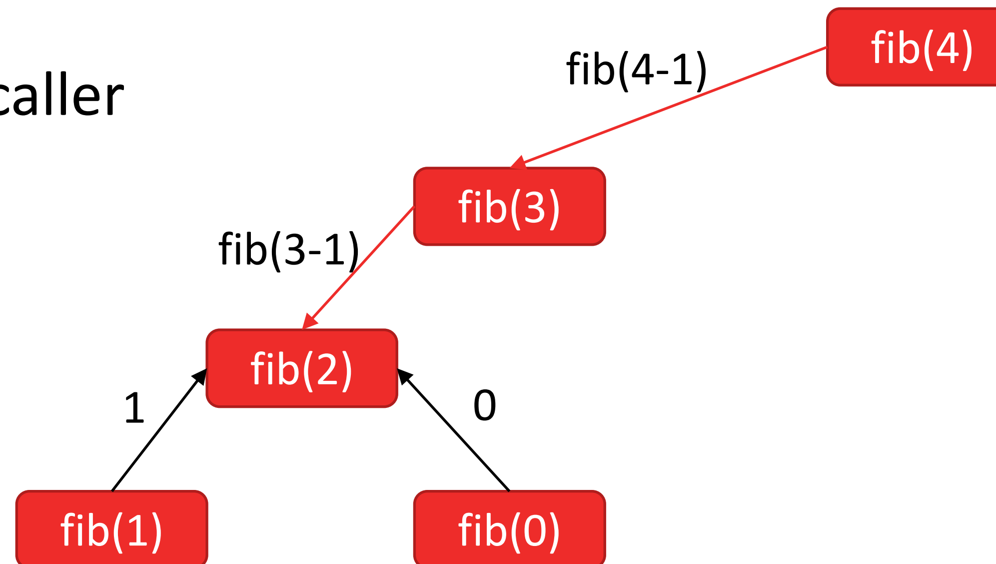
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(2) → 1+0 = 1

Now, fib(2) returns 1 to the caller



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Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

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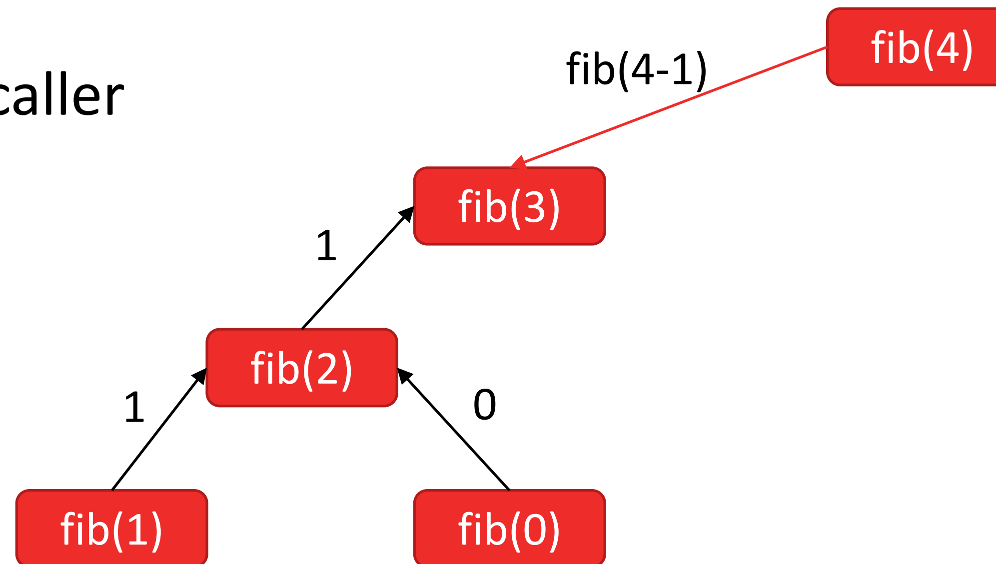

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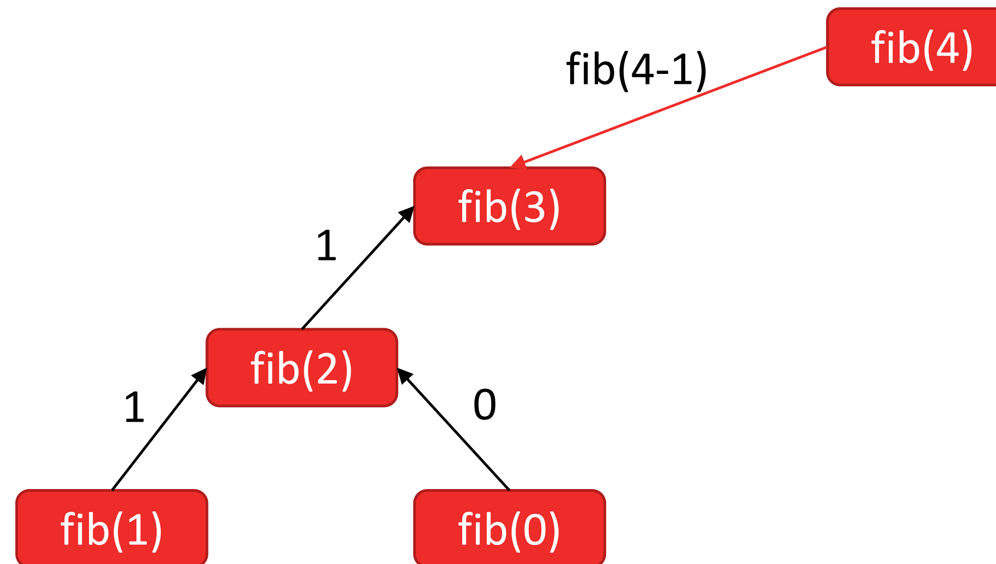
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(3) → 1 + **fib(1)**

Now, fib(1) is calculated



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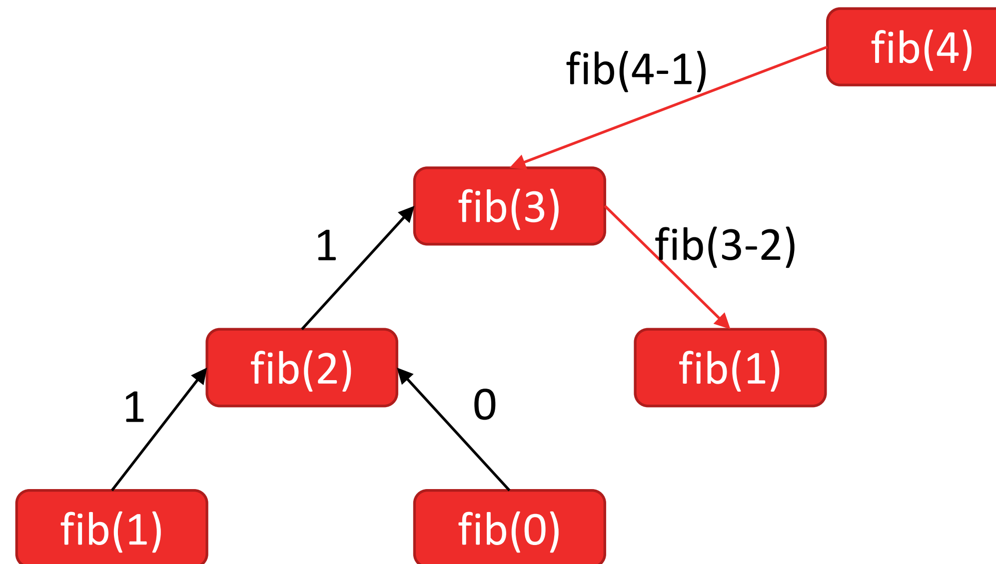
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(1)



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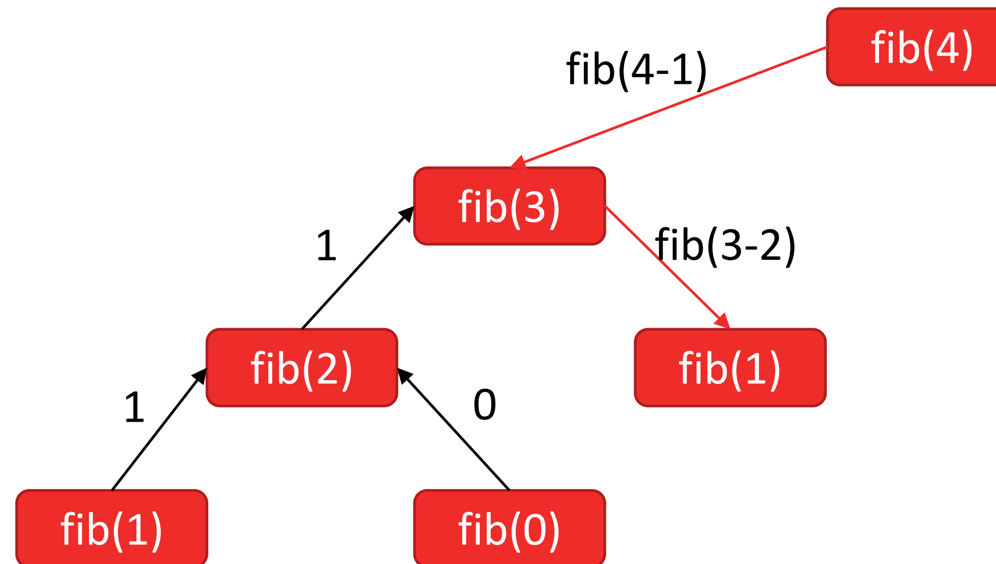
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(1) → 1

fib(1) returns 1 to the caller.



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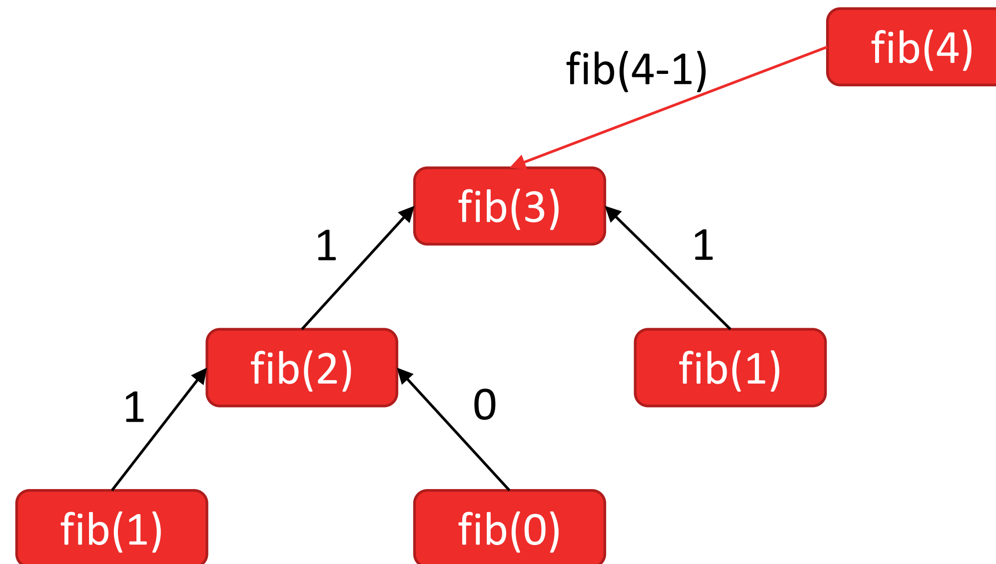
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(1) → 1

fib(1) returns 1 to the caller.



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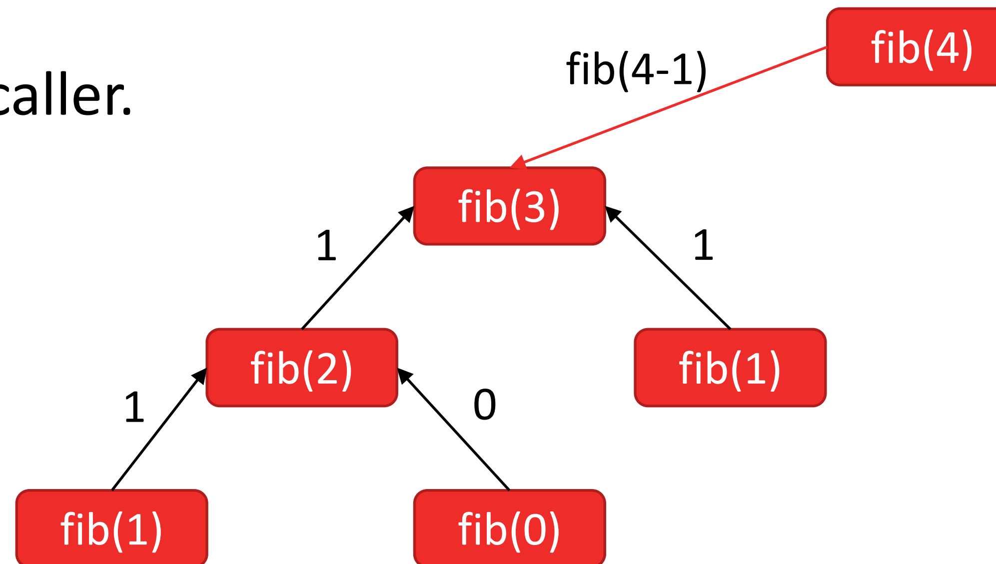
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(3) → 1 + 1 = 2

Now, fib(3) returns 2 to the caller.



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Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

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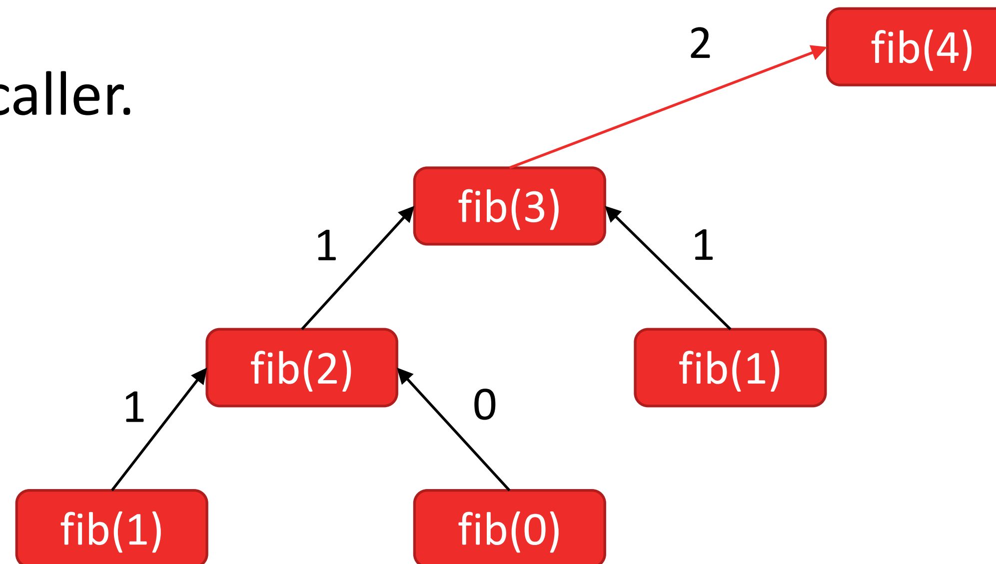

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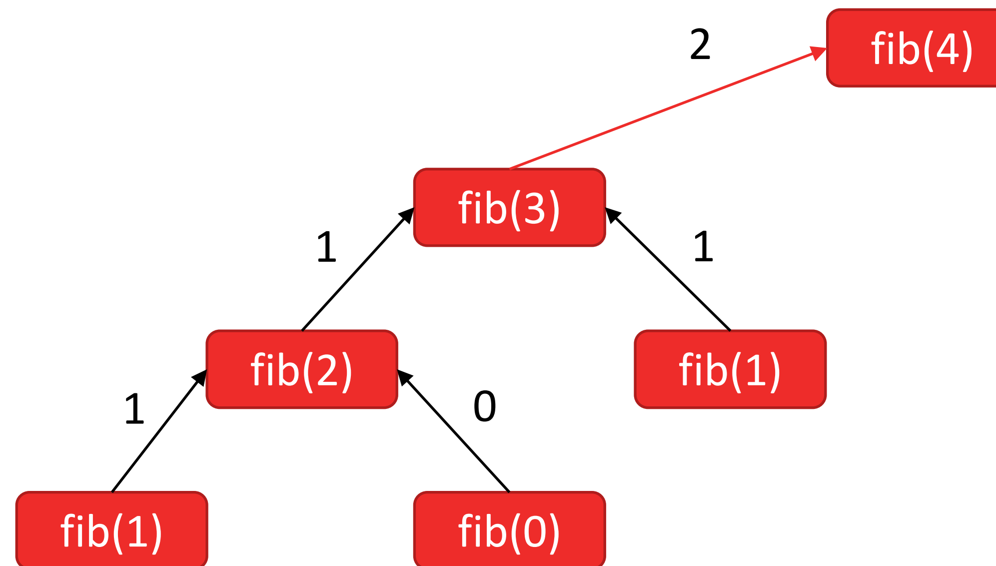
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(4) → 3 + fib(2)



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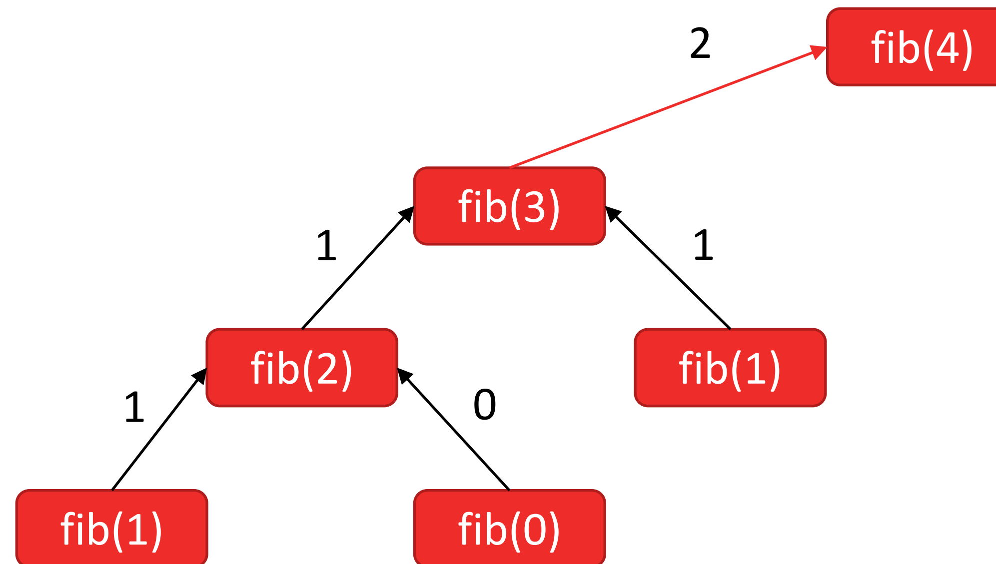
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(4) → 3 + **fib(2)**

fib(2) is calculated.



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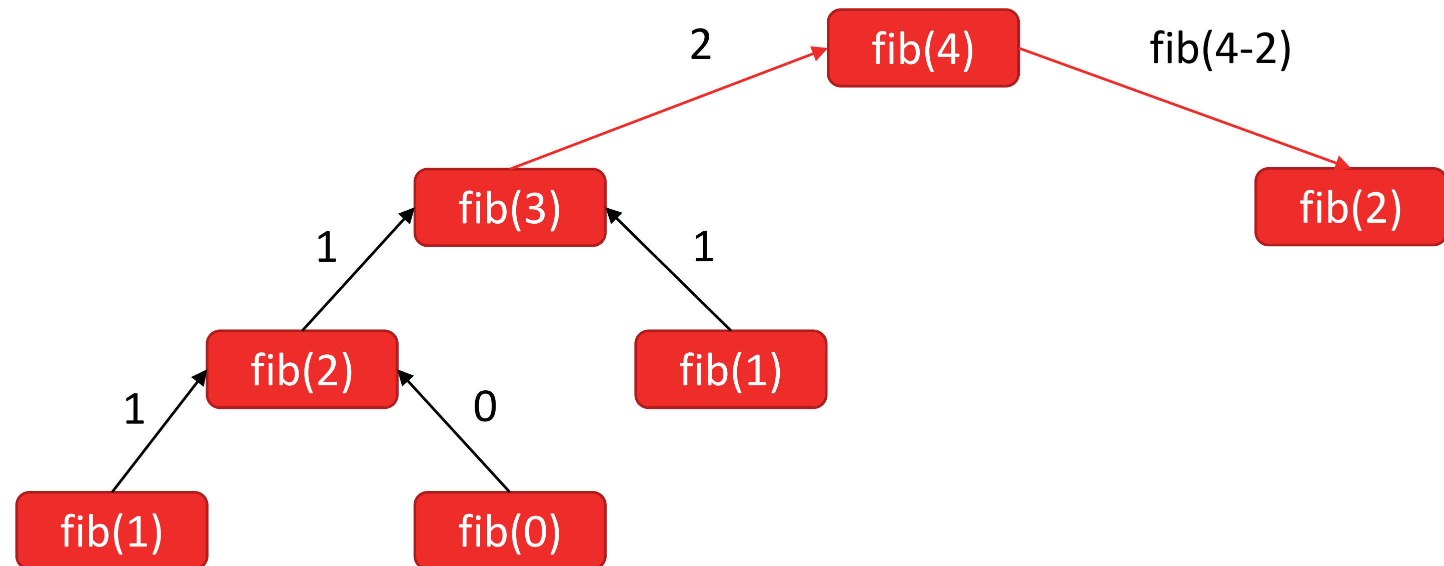
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(2)



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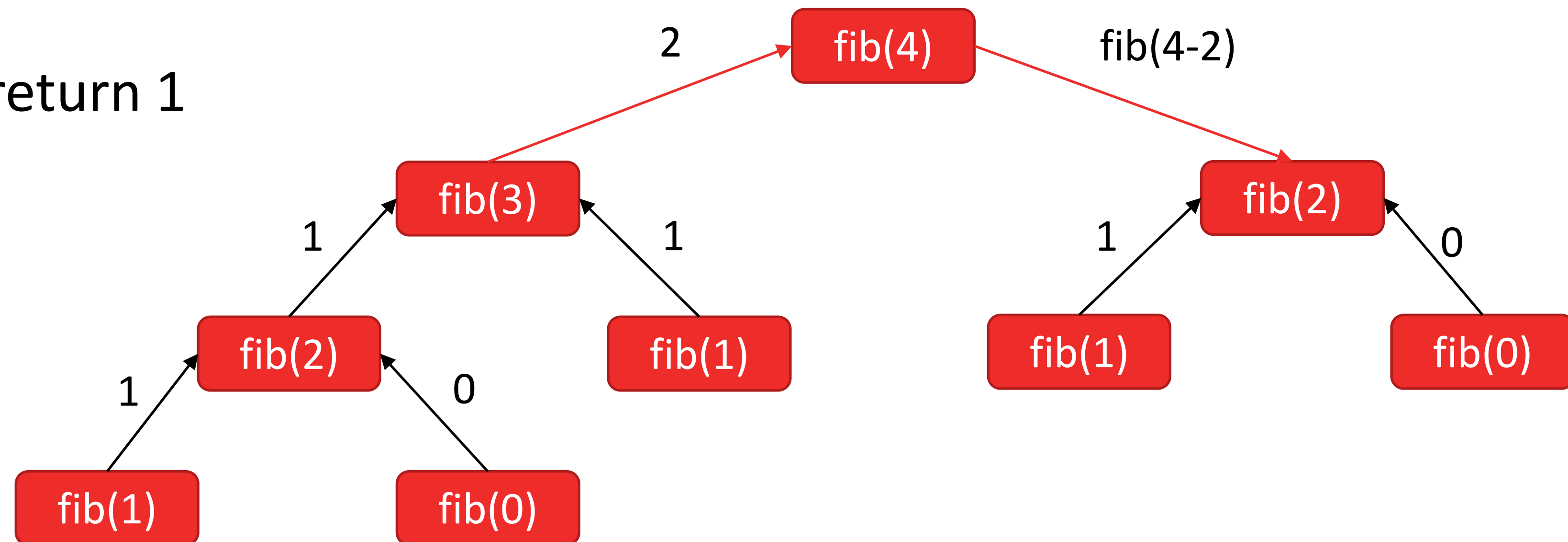
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(2) → **fib(1)** + **fib(0)**

Similar to before, fib(2) will return 1



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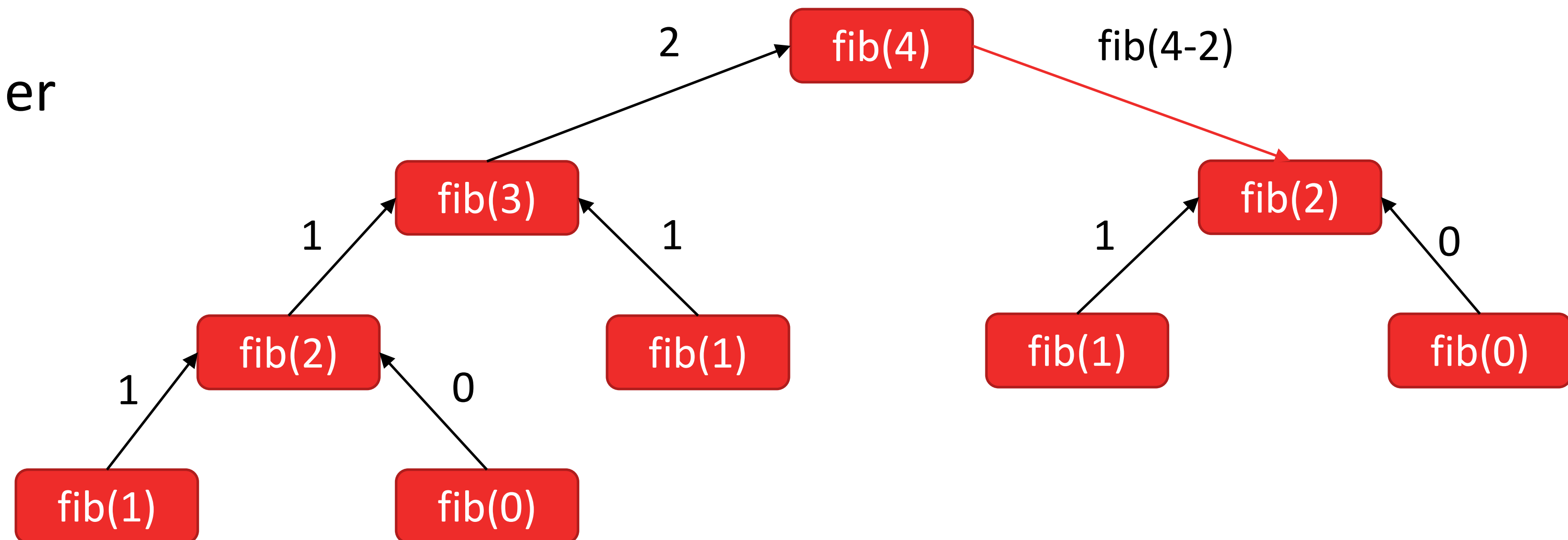
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(2) → 1

fib(2) will return 1 to the caller



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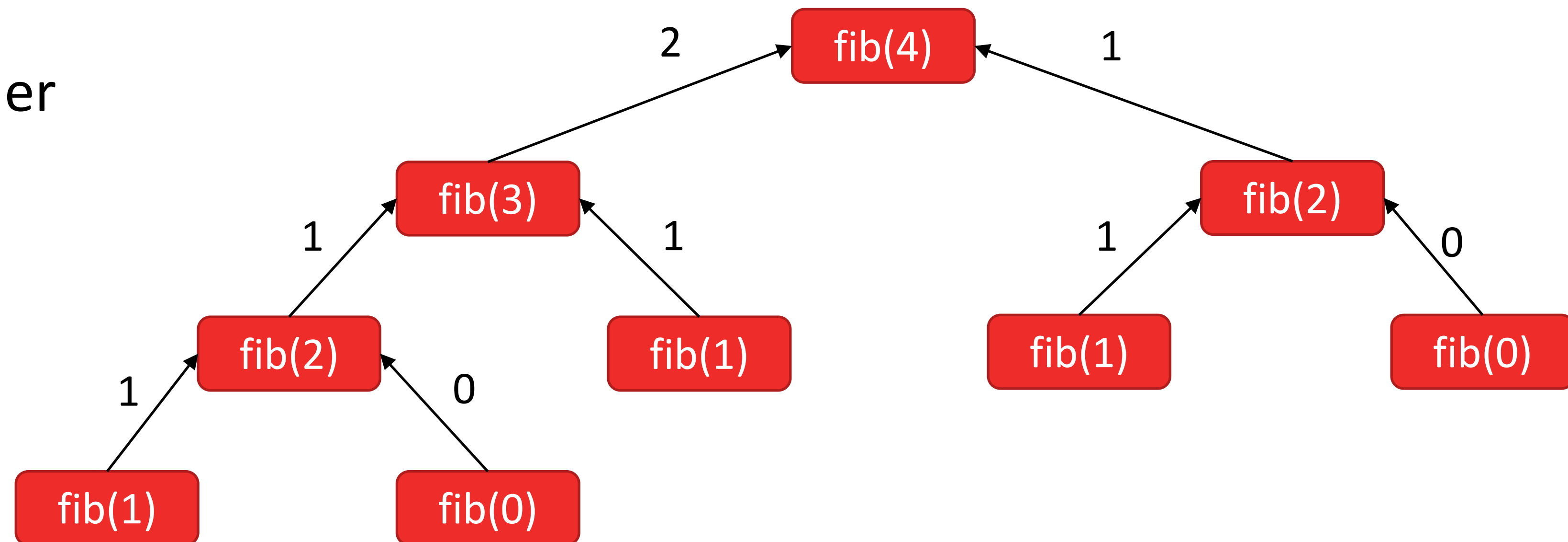
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(2) → 1

fib(2) will return 1 to the caller



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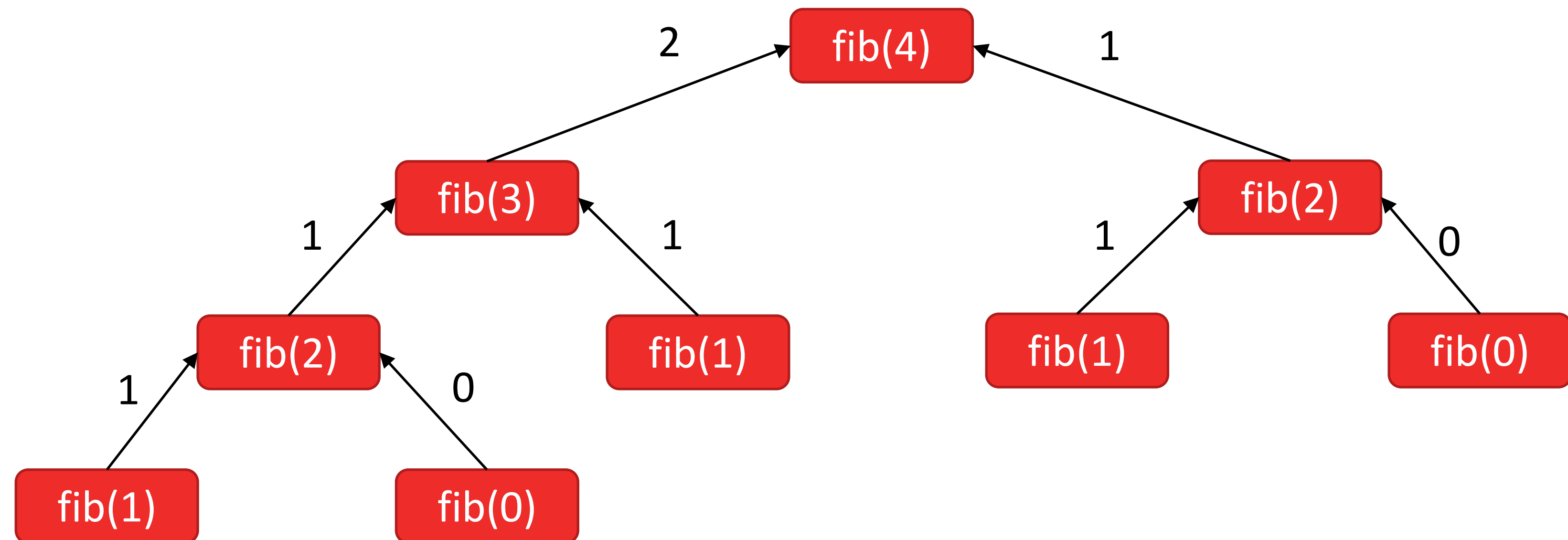
Example - Fibonacci Numbers - Visualization

$$fib(n) = \begin{pmatrix} 0 & n = 0 \\ 1 & n = 1 \\ fib(n-1) + fib(n-2) & n > 1 \end{pmatrix}$$

fib(4) → 2 + 1 = 3

Answer is 3.



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Advantages

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Advantages of Recursion

- Very well suited to some problems
  - Tree traversals
  - Flood fill
  - Fractal images
  - Quick sort / merge sort
  - …
- Often easier to implement, sometimes faster than iterative

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Notice Anything About Fibonacci Numbers

- ?

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Notice Anything About Fibonacci Numbers

- We called fib(1), fib(0) many many many times
- Why not store this information once

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Dynamic Programming

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Dynamic Programming

n=	0	1	2	3	4	5	6	7	…
fib(n)	0	1	1	2	3	5	8	13	

```
def fibDyn(n):
    fib = [0, 1]
    for i in range(2,n+1):
        fib.append(fib[i-2]+fib[i-1])
    return fib[n]
```


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Advantages of Dynamic Programming

- Dynamic programming is recursive in concept
  - but we store solutions we already found!
- We start bottom up instead of top down
  - Generally we find ourselves calculating many solutions at the bottom we find again and again
  - Let’s calculate it once and store it
- Generally faster than similar recursive solution
  - We don’t have to push information on the stack with recursion calls (we store our own state instead)

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Advantages of Dynamic Programming

- Tricky to implement (especially for more complex problems see CPSC 413)
- But not always faster, or guaranteed to take less space than similar recursive solution
  - Sometimes we don’t need need to track all this state, recursion lets us dispose of information stored on stack as the function call returns

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Iterative Programming

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Iterative Programming

- Sometimes we can notice in dynamic programming that we didn’t need to store all the previous function calls
- Or we can noticed that each function call was base iteratively on the previous function calls

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Iterative Programming

```
def fibIter(n):
    if n == 0:
        return 0
    if n == 1:
        return 1
    prev1 = 1
    prev2 = 0
    result = None
    for i in range(2,n+1):
        result = prev1 + prev2
        prev2 = prev1
        prev1 = result
    return result
```


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Advantages of Iteration

- Typically
  - Faster (but not always)
  - Requires less memory (most of the time)

- But some problems are messy to express iteratively

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Fractals

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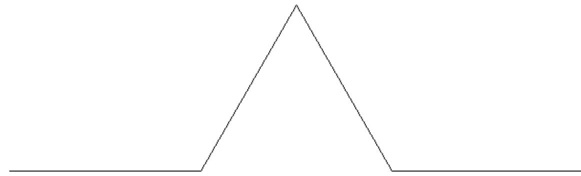
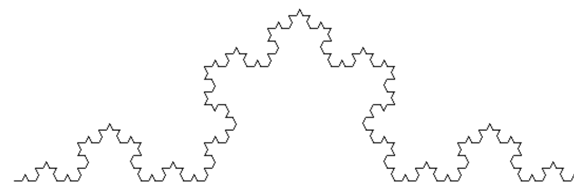
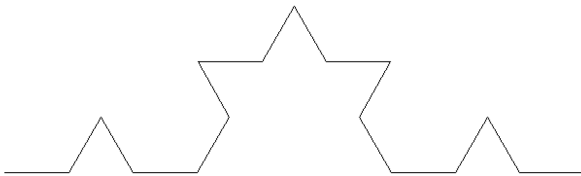
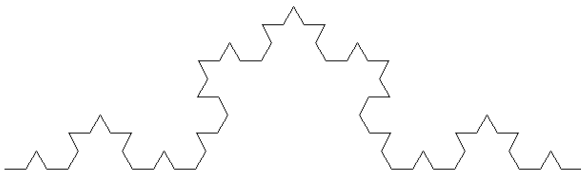
Fractals

- Self similar images
- Often have reasonably simple recursive definitions
  - 1
  - 2
  - 3
  - 4









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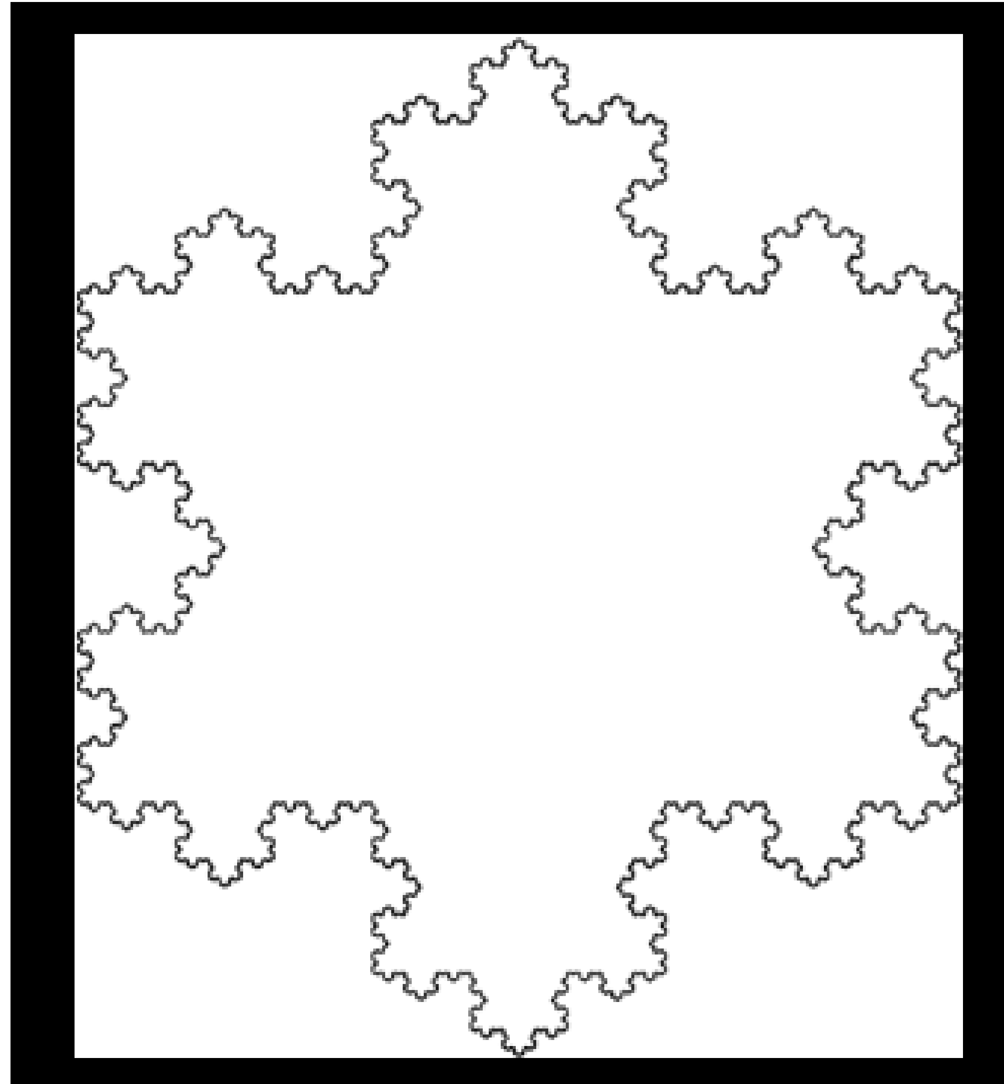
Koch Snowflake



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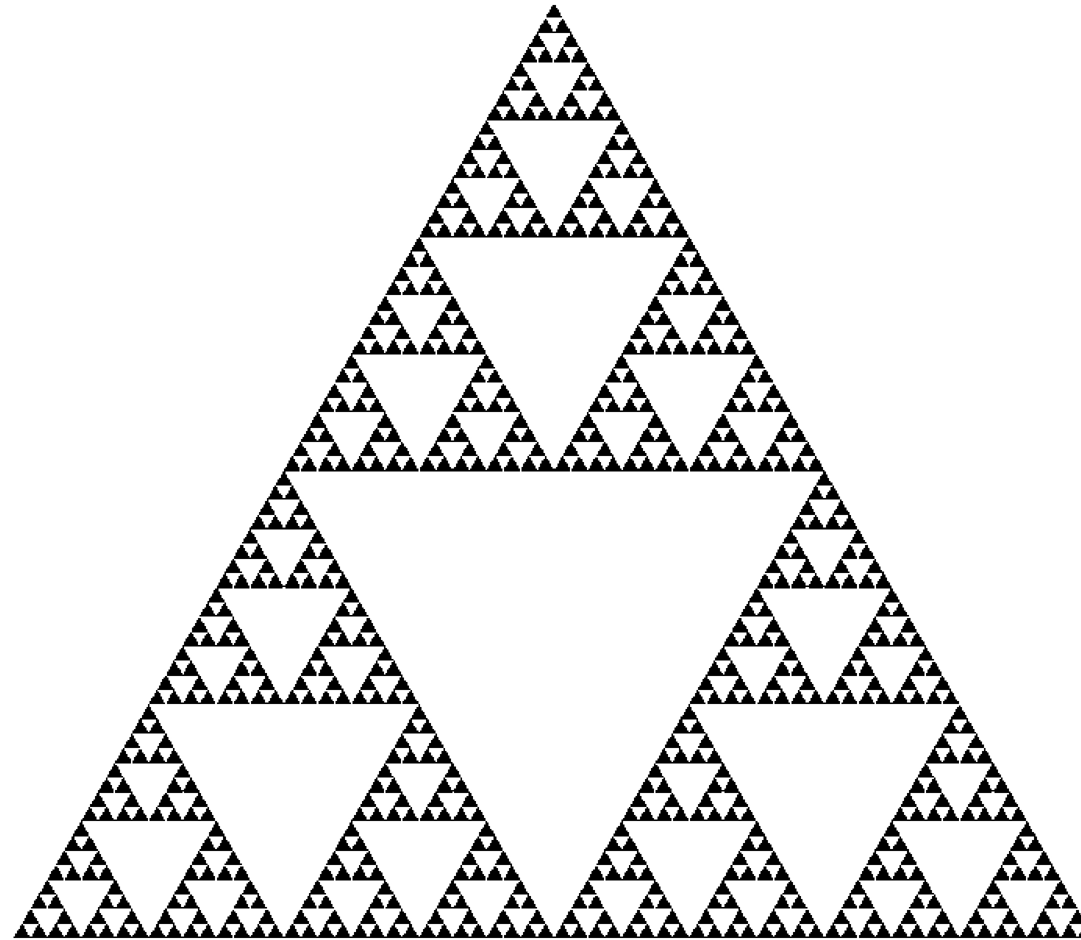
Sierpinski Triangle



Sierpinski Trianglge
Source: http://commons.wikimedia.org/wiki/File:Sierpinski-Trigon-7.svg
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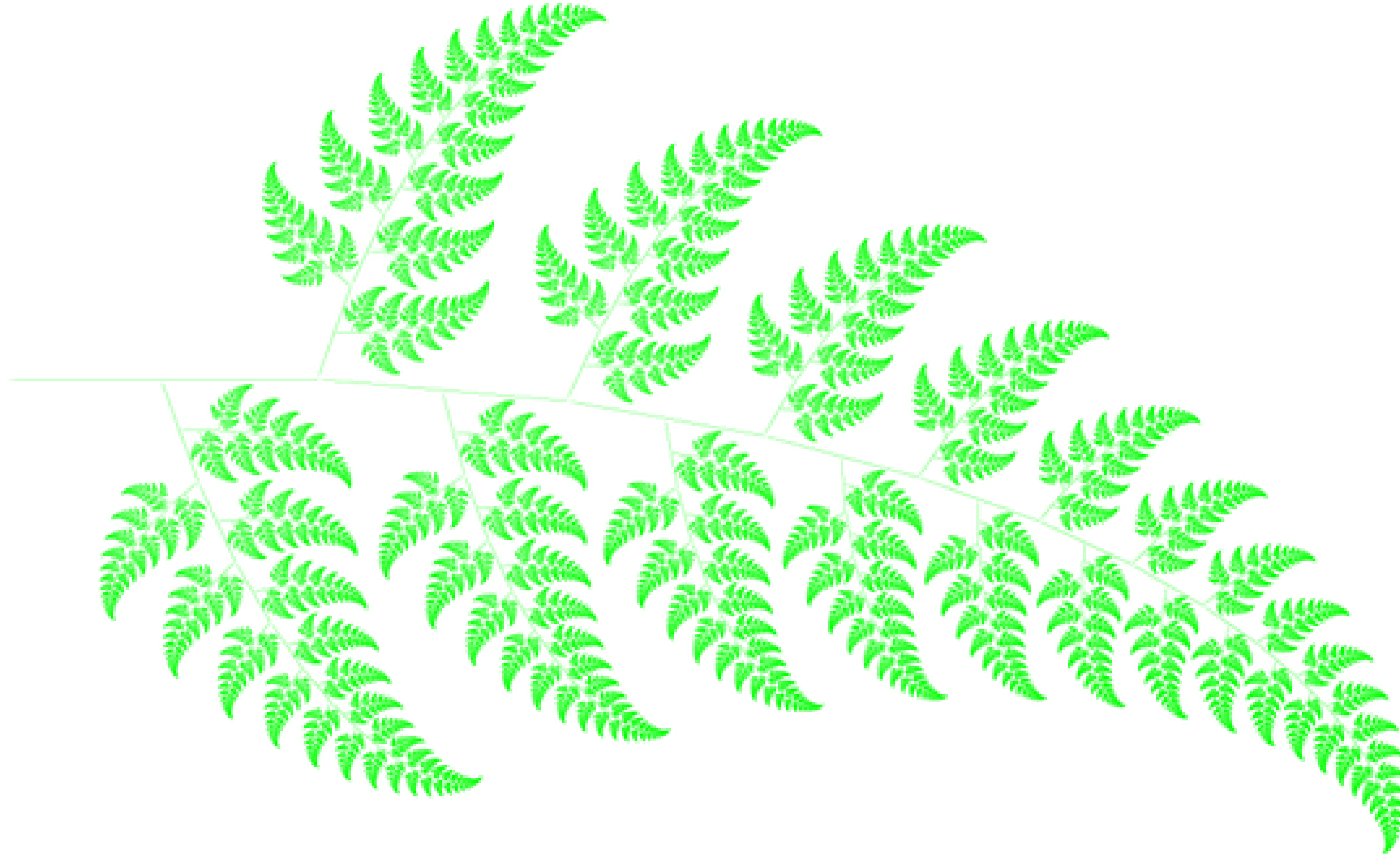
Fractal Fern



Fractal Fern
Source: http://schools-wikipedia.org/images/67/6740.png.htm
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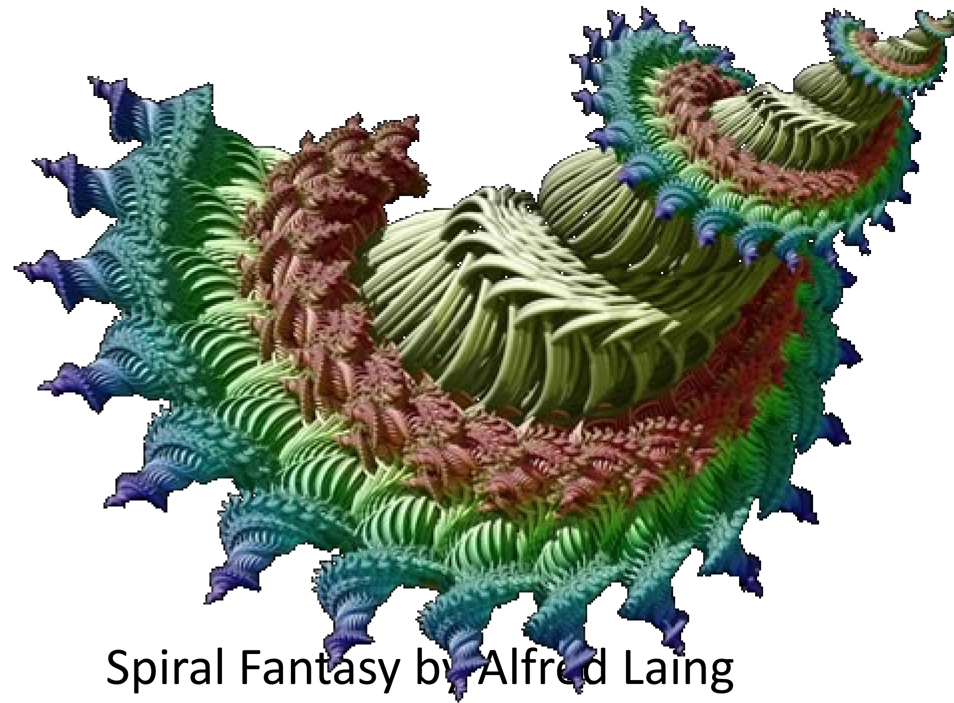
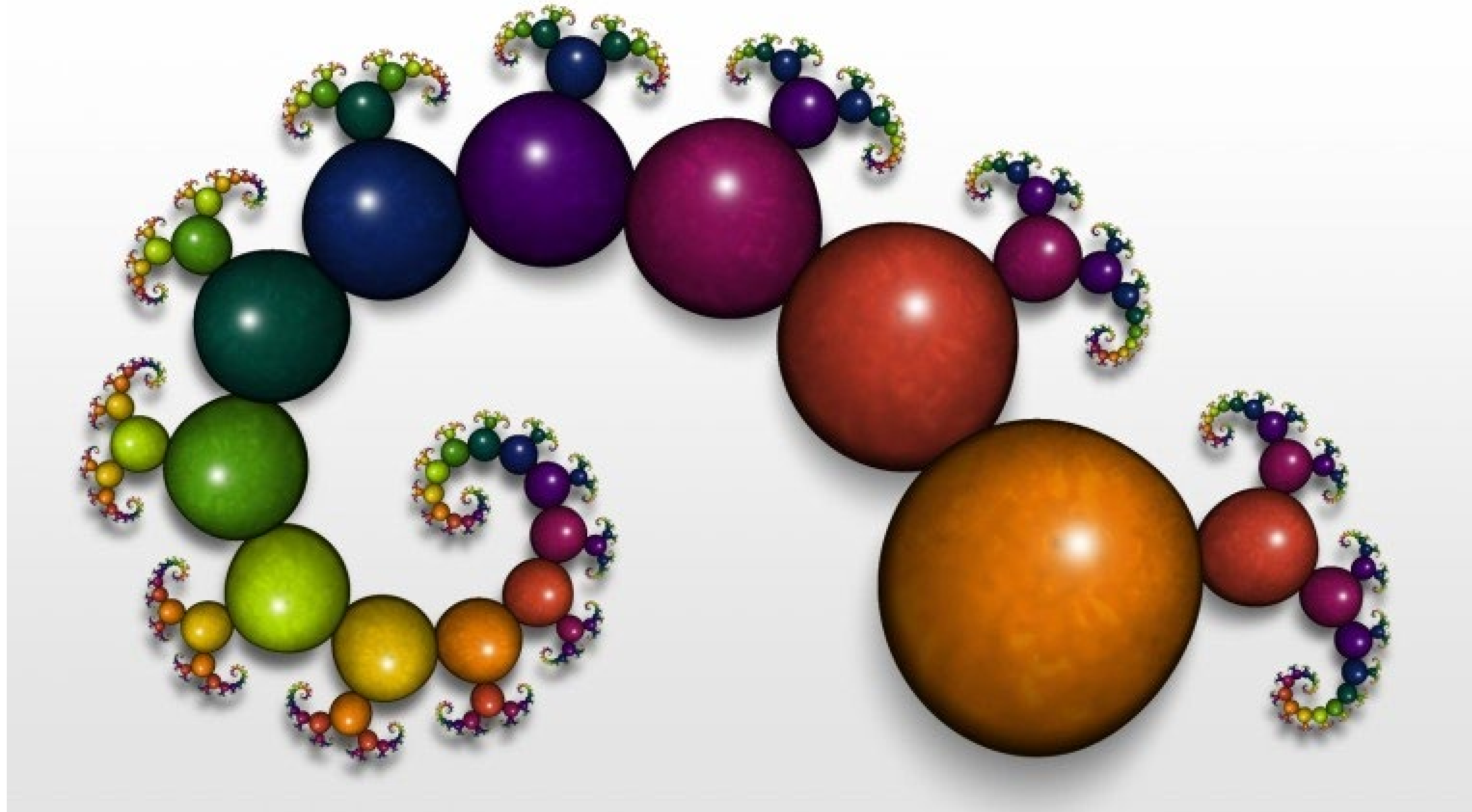
Fractal Art



Spiral Fantasy by Alfred Laing

Fractal Art



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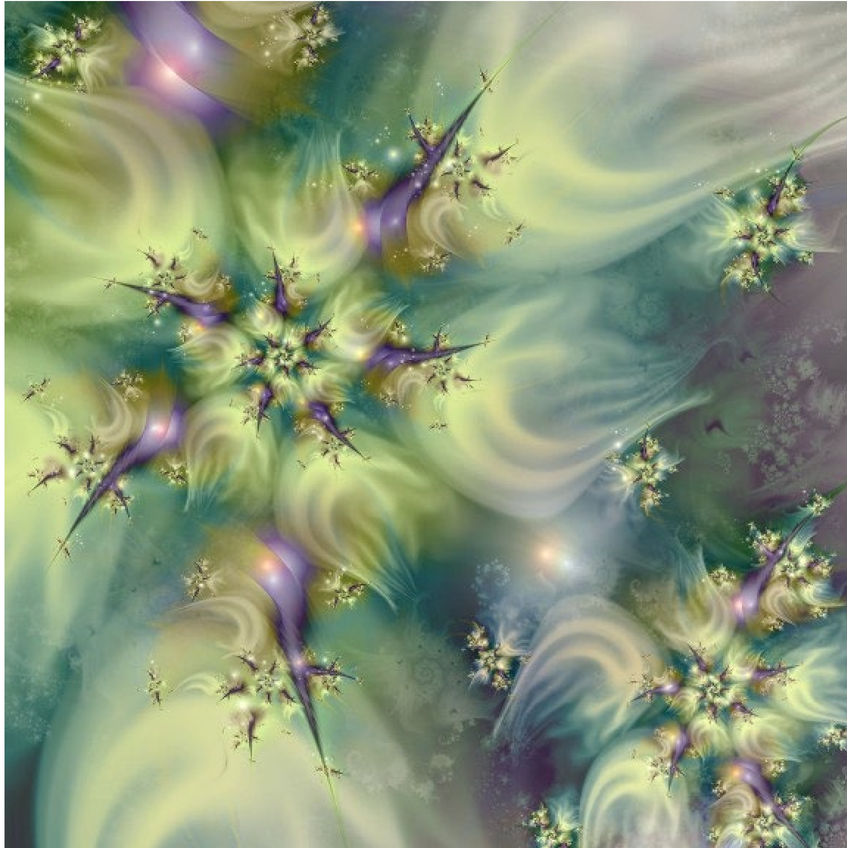
Fractal Art



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Rose by Keith Mack

Mazes

Maze Path Finding

- Consider a two dimensional list containing 4 different values
  - Entrance for the maze
  - Exit for the maze
  - Open spaces
  - Walls
- Assume that the maze is fully enclosed

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Maze Path Finding

- Algorithm solve(map, x, y)
  - If the current square is a wall or a space we have already visited, return failure
  - If the current square is the exit point, mark it as part of the solution and return success
  - Mark the current square as part of the solution

  - If solve(map, x, y+1) is successful, return success
  - If solve(map, x, y-1) is successful, return success
  - If solve(map, x+1, y) is successful, return success
  - If solve(map, x-1, y) is successful, return success

  - Mark the current square as visited but not part of the solution
  - Return failure

Maze Path Finding

```
def solve(map,x,y):
    if map[x][y] == "wall" or map[x][y] == "visited":
        return False

    if map[x][y] == "exit":
        print "Go to (%d,%d)" % (x,y)
        return True
```


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Maze Path Finding

```
print “Go to (%d,%d)” % (x,y)
if solve(map,x,y+1):
   return True
elif solve(map,x,y-1):
   return True
elif solve(map,x+1,y):
   return True
elif solve(map,x-1,y):
   return True
else:
   map[x][y] = “visited”
   return False
```


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Recursion – Beyond Algorithms

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Why recursion

- Recursive implementation is the perfect fit for naturally recursive problems.
- The result of one step depends on the previous steps.
- There is a base case where the process will stop at.
- e.g., search, games, various math problems
- Computationally elegant
- Easy to program than using loops
- Tradeoffs:
  - Function calls can be costly in terms of memory and time

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Induction

- Recursive functions performance and even their results are often shown via proofs using induction
  - Base Case: Where recursion bottoms out and returns
  - Inductive Case: Where recursion calls another functions call

- Induction is generally be taught in CPSC 271, 331, and 413

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Example

```
def binary_search(t_list, k):
    if len(t_list) != 0:
        center = len(t_list)//2
        if t_list[center] == k:
            return True
        if t_list[center] > k:
            return binary_search(t_list[:center], k)
        else:
            return binary_search(t_list[center+1:], k)
    return False
```


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Onward to … CPSC 219.

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